Lefschetz for non-Archimedean Jacobians

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Introduction

Let Y be a smooth projective variety of dimension n over a non-Archimedean field K, and let D be an ample divisor on Y. As a consequence of comparison theorems of Berkovich and of the ℓ-adic Lefschetz hyperplane theorem, the inclusion $D^{an} \hookrightarrow Y^{an}$ induces isomorphisms between the $\mathbb{Q}$-cohomology groups of dimensions $< n-1$, and an injection in dimension $n-1$.

However, unlike the case of complex analytic spaces, this does not hold in general if we replace $\mathbb{Q}$ by $\mathbb{Z}$, or if we replace cohomology groups with homotopy groups.

Question. *For the analytifications of specific families of projective varieties, is it possible to establish a Lefschetz hyperplane theorem for $\mathbb{Z}$-cohomology and homotopy groups?*

Let X be a smooth projective curve over K of genus g, and let J be its Jacobian. Suppose K is algebraically closed, and choose an identification $\mathrm{Pic}^d(X) \cong J$. Let W_d denote the locus in J of effective divisor classes of degree d.

Theorem 1 (S, 2016)**.** *For $1 \leq d \leq g-1$, the inclusion $W_d^{an} \hookrightarrow J^{an}$ induces isomorphisms between $\mathbb{Z}$-cohomology (resp. homotopy) groups of dimensions $< d$, and an injection (resp. surjection) in dimension d.*

Recall that W_{g-1} is equal, up to translation, to the theta divisor Θ of J. We thus obtain, as a special case of the above theorem, a Lefschetz hyperplane theorem for $\mathbb{Z}$-cohomology and homotopy groups for the pair (J^{an}, Θ^{an}).

Tropicalization

To prove Theorem 1, we tropicalize W_d^{an} and show that this tropicalization is a homotopy equivalence. Recall that X^{an} has a tropicalization Γ (see Figure 1), which is a metric graph. There is a natural tropicalization map from J^{an} to the Jacobian $J(\Gamma)$ of Γ, which sends W_d^{an} to a subset $W_d(\Gamma)$ of $J(\Gamma)$.

Theorem 2 (S, 2016)**.** *The map $W_d^{an} \to W_d(\Gamma)$ is a homotopy equivalence for all d.*

The tropicalization map $\mathrm{trop} : J^{an} \to J(\Gamma)$ is naturally identified with the map from $\mathrm{Pic}^{d,an}(X)$ to $\mathrm{Pic}^d(\Gamma)$ given by tropicalizing the divisors on X. In particular, this allows us to identify $W_d(\Gamma)$ with the locus in $\mathrm{Pic}^d(\Gamma)$ of effective tropical divisor classes of degree d.

Morphisms with Projective Fibers

The natural map $\mathrm{Sym}^d(\Gamma) \to W_d(\Gamma)$ has contractible fibers, and therefore is a homotopy equivalence. Theorem 2 can then be reduced to showing that the map $\mathrm{Sym}^{d,an}(X) \to W_d^{an}$ is a homotopy equivalence. This is derived as a special case of the following.

Theorem 3 (S, 2016)**.** *Let $f : Z \to Y$ be a morphism of quasi-projective K-varieties. Suppose that there is a finite stratification $Y = \coprod_i Y_i$ such that $f : Z \times_Y Y_i \to Y_i$ is a projective bundle of rank r_i over Y_i.*
Then, there is a finite extension $K \subset L$ such that $f_L^{an} : (Z_L)^{an} \to (Y_L)^{an}$ is a homotopy equivalence.

Brown and Foster have shown, following the MMP approach developed by Mustaţă – Nicaise and Nicaise – Xu, that over $K = \mathbb{C}((t))$, if $f : Z \to Y$ is a projective bundle with Y smooth, then $f^{an} : Z^{an} \to Y^{an}$ is a homotopy equivalence.

Our proof of Theorem 3 follows a different approach, which works over K of arbitrary characteristic and allows arbitrarily bad singularities for Y. A crucial component of our argument is the theorem of Hrushovski – Loeser – Poonen stating that over K with a countable dense subset, the analytification Y^{an} of a quasi-projective K-scheme of dimension d embeds into $\mathbb{R}^{2d+1}$. In particular, Y^{an} is metrizable and has a countable dense subset, which allows us to apply the Vietoris-Begle-Smale mapping theorem for maps with contractible fibers.

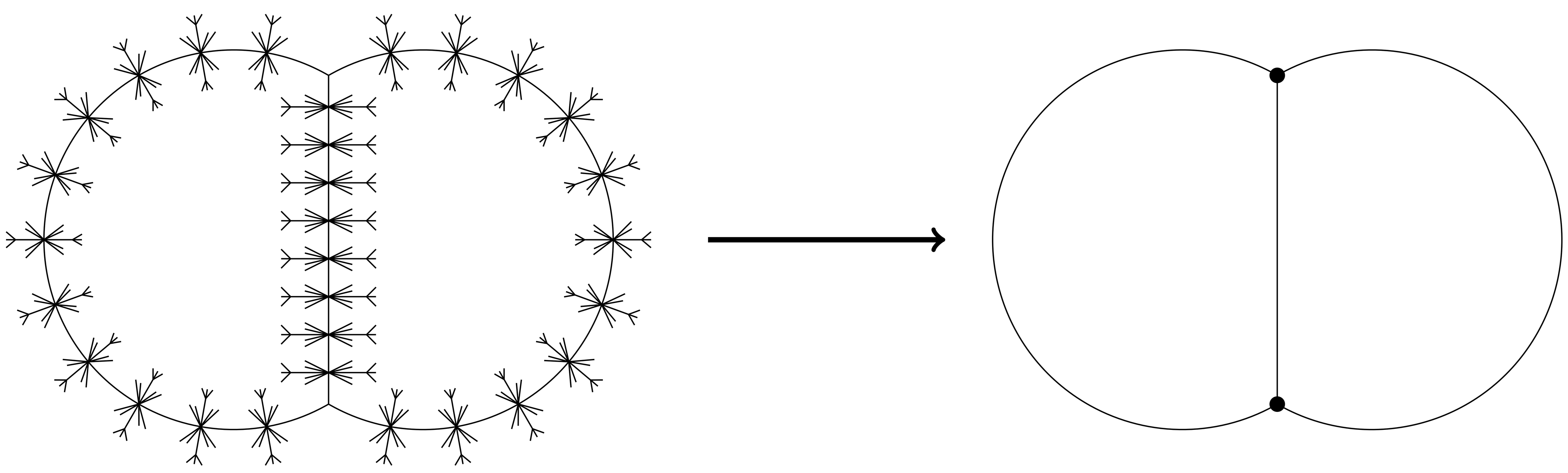

Figure 1: The tropicalization of an analytic curve X^{an}.

We thank D. Ranganathan for permission to use the picture on the left.