Weighted Hurwitz numbers and topological recursion I.

J. Harnad

Centre de recherches mathématiques
Université de Montréal

Department of Mathematics and Statistics
Concordia University

IHP trimester: Combinatorics and interactions
Workshop: Enumerative geometry and combinatorics of moduli spaces. March 13 - 17 , 2017

*Based in part on joint work with: M. Guay-Paquet, A. Yu. Orlov, A. Alexandrov, G. Chapuy, B. Eynard

1. **General classical Hurwitz numbers**
   - Enumerative group theoretical meaning (Frobenius)
   - Geometric meaning (Hurwitz)
   - Graphical encoding of branched covers "*Constellations*"
   - Simple double Hurwitz numbers (Okounkov/Pandharipande)
2. **KP and 2D Toda τ-functions as generating functions**
   - Simple (single and double) Hurwitz numbers
   - τ-functions as generating functions for Hurwitz numbers
   - Weighted Hurwitz numbers: weighted branched coverings
   - Geometric weighted Hurwitz numbers: weighted coverings
   - Combinatorial weighted Hurwitz numbers: weighted paths
   - Examples
3. **Fermionic representations and topological recursion**
   - Fermionic representation of τ-function and Baker function
   - Fermionic representation of adapted basis
   - Spectral curve: quantum and classical
   - Pair correlator (spectral kernel)
   - Current correlators as generating functions

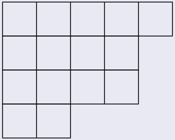
Factorization of elements in S_N

Question: What is the number $N!F(\mu^{(1)}, \dots \mu^{(k)}, \mu)$ of distinct ways the identity element $\mathbf{I} \in S_N$ in the symmetric group S_N can be written as a product

$$\mathbf{I} = h_1 h_2 \cdots h_k$$

of k elements $h_i \in S_N$ in the conjugacy classes of cycle type $h_i \in \mathrm{cyc}(\mu^{(i)})$ for a given sequence of partitions $\{\mu^{(i)}\}_{i=1,\dots,k}$ of N?

Young diagram of a partition. Example $\mu = (5, 4, 4, 2)$



Representation theoretic answer (Frobenius-Schur)

The **Frobenius-Schur** formula expresses this in terms of characters:

$$F(\mu^{(1)}, \dots \mu^{(k)}) = \sum_{\lambda, |\lambda|=N} h_\lambda^{k-2} \prod_{i=1}^{k} \frac{\chi_\lambda(\mu^{(i)})}{z_{\mu^{(i)}}}, \quad |\mu^{(i)}| = N$$

where $h_\lambda = \left(\det \frac{1}{(\lambda_i - i + j)!}\right)^{-1}$ is the **product of the hook lengths** of the partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_{\ell(\lambda} > 0$, where $\chi_\lambda(\mu^{(i)})$ is the **irreducible character** of representation λ evaluated in the conjugacy class $\mu^{(i)}$, and

$$z_\mu := \prod_i i^{m_i(\mu)}(m_i(\mu))! = |\mathrm{aut}(\mu)|$$

is the **order of the stabilizer** of an element of $\mathrm{cyc}(\mu)$ **(**$m_i(\mu) =$**# parts** μ_j **of** μ **equal to** i**)**.

Geometric meaning (Hurwitz)

Hurwitz numbers: Let $H(\mu^{(1)}, \dots, \mu^{(k)})$ be the number of inequivalent branched N-sheeted covers of the Riemann sphere, with k branch points, and ramification profiles $(\mu^{(1)}, \dots, \mu^{(k)})$ at these points.

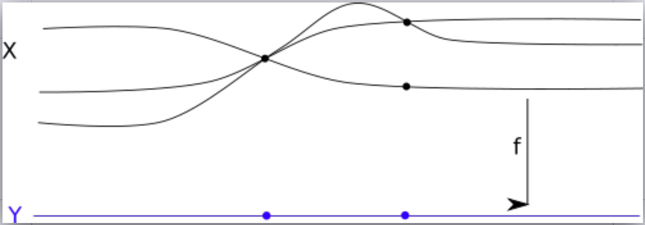
The **Euler characteristic** of the covering curve is given by the **Riemann-Hurwitz formula**:

$$2 - 2g = 2N - d, \quad d := \sum_{i=1}^{l} \ell^*(\mu^{(i)}),$$
$$g = \text{genus of covering curve},$$

where $\ell^*(\mu) := |\mu| - \ell(\mu) = N - \ell(\mu)$ is the **colength** of the partition. The **Monodromy Representation** shows these two enumerative invariants are identical.

$$H(\mu^{(1)}, \dots, \mu^{(k)}) = F(\mu^{(1)}, \dots, \mu^{(k)}).$$

Example: 3-sheeted branched cover with ramification profiles (3) and $(2, 1)$



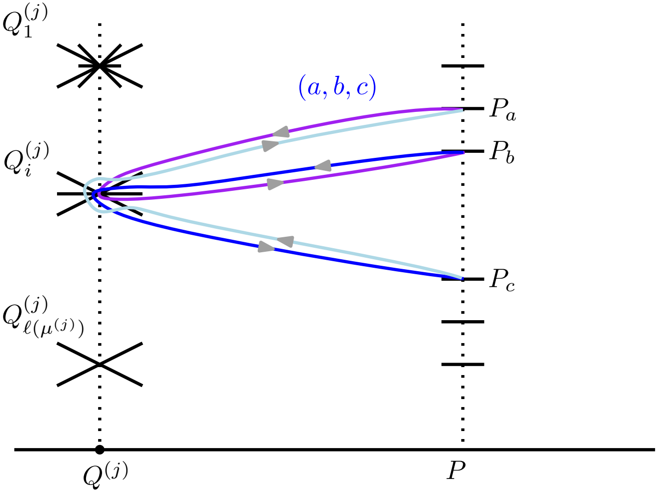
Graphical encoding of branched covers: *Constellations*

Constellations:

1. Let P be a generic (non-branched) base point of the covering $\mathcal{C} \to \mathbf{CP}^1$ and $(P_1, \dots, P_N)$ an ordering of the points of $\mathcal{C}$ over P.
2. Let $(Q^{(1)}, \dots Q^{(k)})$ be an ordering of the branch points of the cover $\Gamma \to \mathbf{CP}^1$, with $(Q_j^{(i)}, j = 1, \dots, \mu_j^{(i)}$ the ramification points over these, having ramification indices $\mathrm{ram}(Q_j^{(i)}) = \mu_j^{(i)}$ equal to the parts of the partitions $(\mu^{(1)}, \dots, \mu^{(k)})$
3. For each simple, closed curve $\mathcal{C}_i$ in $P \in \mathbf{CP}^1$ based at $P \in \mathbf{CP}^1$ and going once around $Q^{(i)}$ in the positive sense, there is a unique lift to Γ whose monodromy is an element $h_i \in \mathrm{cyc}(\mu^{(i)} \subset S_N$

Constellations (cont'd)

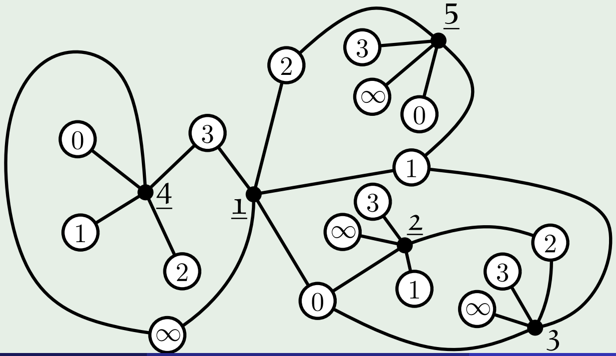
4. Draw a bipartite graph on Γ, with vertices of two types: "coloured" and "stars". The "coloured" vertices are the points $\{Q_j^{(i)}\}$ and the star vertices are the point $\{P_1, \dots, P_N\}$. The edges consist of the pairs of segments of the contours around each ramification point $Q_j^{(i)}$ starting or ending at one of the unramified ones $P_1, \dots, P_N$.



To these, we add two more “fixed” branch points, say $Q^{(0)} = 0$, $Q^{(k+1)} = \infty$, with ramification profiles $\mu^{(0)} := \mu$ and $\mu^{(k+1)} := \nu$

Example ($N = 5, \quad k = 3$)

$$h_1 = (135), \quad h_2 = (15)(23), \quad h_3 = (14),$$
$$h_0 = (321), \quad h_4 = h_\infty = (14)$$
$$\mu^{(1)} = (3, 1, 1)), \quad \mu^{(2)} = (2, 2, 1), \quad \mu^{(3)} = (2, 1, 1, 1),$$
$$\mu := \mu^{(0)} = (3, 1, 1)), \quad \nu := \mu^{(4)} = (2, 1, 1, 1)$$



Example: Simple single/double Hurwitz numbers (Pandharipande/Okounkov)

In particular, choosing only simple ramifications $\mu^{(i)} = (2, (1)^{n-2})$ at $d = k$ points and one further arbitrary one μ at a single point, say, 0, we have the **single simple Hurwitz number**:

$$H^d(\mu) := H((2, (1)^{n-1}), \dots, (2, (1)^{n-1}), \mu).$$

By the **Frobenius-Schur formula** this is

$$H^d(\mu) = \sum_{\lambda, |\lambda| = |\mu|} \frac{\chi_\lambda(\mu)}{z_\mu h_\lambda} (\mathrm{cont}_\lambda)^d,$$

where the **content sum** of the Young diagram associated to λ is defined as

$$\mathrm{cont}(\lambda) := \sum_{(ij) \in \lambda} (j - i) = \frac{1}{2} \sum_{i=1}^{\ell(\lambda)} \lambda_i(\lambda_i - 2i + 1) = \frac{\chi_\lambda((2, (1)^{n-2}) h_\lambda}{z_{(2,(1)^{n-2})}}$$

Simple single/ double Hurwitz numbers (Pandharipande/Okounkov)

The **simple (double) Hurwitz number** (Okounkov (2000)), defined as

$$\mathrm{Cov}_d(\mu,\nu) = H^d_{\exp}(\mu,\nu)) := H((2,(1)^{n-1}),\dots,(2,(1)^{n-1}),\mu,\nu)$$

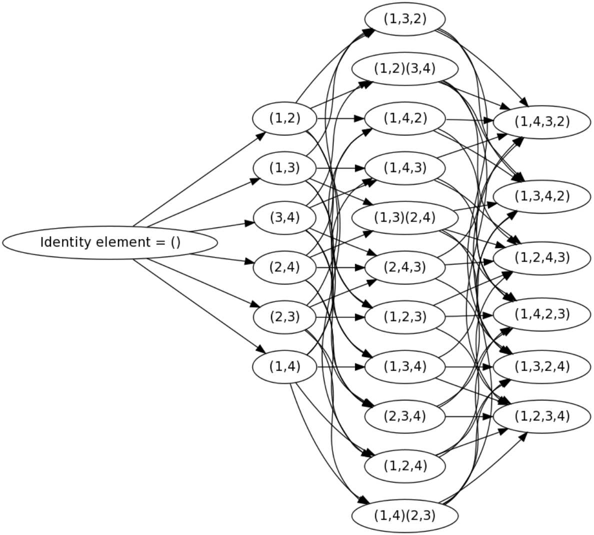
have the ramification types (μ,ν) at two points, say $(0,\infty)$, and simple ramification $\mu^{(i)} = (2,(1)^{n-2})$ at $d = k$ other branch points.

Combinatorial meaning: paths in the Cayley graph

Combinatorially, this equals the number of d-step paths in the **Cayley graph** of S_n generated by **transpositions**, starting at an element $h \in \mathrm{cyc}(\mu)$ and ending in the conjugacy class $cyc(\nu)$.

Example: Cayley graph for S_4 generated by all transpositions

Transpositioncayleyons4.png 867×779 pixels 14-08-23 10:17 PM



Hypergeometric τ-function as generating function for simple single and double Hurwitz numbers: (Okounkov, Pandharipande)

Define

$$\tau^{mKP(\gamma,\beta)}(N,\mathbf{t}) := \sum_{\lambda} \gamma^{|\lambda|} r_\lambda^{\exp}(N,\beta)) h_\lambda^{-1} s_\lambda(\mathbf{t})$$

$$\tau^{2DToda(\gamma,\beta)}(N,\mathbf{t},\mathbf{s}) := \sum_{\lambda} \gamma^{|\lambda|} r_\lambda^{\exp}(N,\beta) s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s})$$

where $\quad r_\lambda^{\exp}(N,\beta) := \prod_{(ij)\in\lambda} r_{N+j-i}^{\exp}(\beta), \quad r_j^{\exp}(\beta) := e^{j\beta}$

and

$$\mathbf{t} = (t_1, t_2, \dots), \quad \mathbf{s} = (s_1, s_2, \dots)$$

are the KP and 2D Toda flow variables.
For $N = 0$, we have

$$r_\lambda^{\exp}(0,\beta) = e^{\beta\,\mathrm{cont}(\lambda)}$$

mKP Hirota bilinear relations for $\tau_g^{mKP}(N, \mathbf{t})$, $\mathbf{t} := (t_1, t_2, \dots)$, $N \in \mathbf{Z}$

$$\oint_{z=\infty} z^{N-N'} e^{-\xi(\delta\mathbf{t},z)} \tau_g^{mKP}(N, \mathbf{t} - [z^{-1}]) \tau_g^{mKP}(N', \mathbf{t} + \delta\mathbf{t} + [z^{-1}]) = 0$$

$$\xi(\delta\mathbf{t}, z) := \sum_{i=1}^{\infty} \delta t_i \, z^i, \quad [z^{-1}]_i := \frac{1}{i} \, z^{-i}, \quad \text{identically in } \delta\mathbf{t} = (\delta t_1, \delta t_2, \dots)$$

$2D$ Toda Hirota bilinear relations for $\tau_g^{2Toda}(N, \mathbf{t}, \mathbf{s})$, $\mathbf{s} := (s_1, s_2, \dots)$

$$\oint_{z=\infty} z^{N-N'} e^{-\xi(\delta\mathbf{t},z)} \tau_g^{2Toda}(N, \mathbf{t} - [z^{-1}], \mathbf{s}) \tau_g^{2Toda}(N', \mathbf{t} + \delta\mathbf{t} + [z^{-1}], \mathbf{s}) =$$

$$\oint_{z=0} z^{N-N'} e^{-\xi(\delta\mathbf{s},z)} \tau_g^{2Toda}(N+1, \mathbf{t}, \mathbf{s} - [z]) \tau_g^{2Toda}(N'-1, \mathbf{t}, \mathbf{s} + \delta\mathbf{s} + [z])$$

$$[z]_i := \frac{1}{i} z^i, \quad \text{identically in } \delta\mathbf{t} = (\delta t_1, \delta t_2, \dots), \ \delta\mathbf{s} := (\delta s_1, \delta s_2, \dots)$$

Change of basis: Frobenius character formula

Using the **Frobenius character formula**:

$$s_\lambda(\mathbf{t}) = \sum_{\mu,\, |\mu|=|\lambda|} \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu(\mathbf{t})$$

where we restrict to

$$it_i := p_i, \quad is_i := p'_i$$

and the p_μ's are the **power sum symmetric functions**

$$p_\mu = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}, \quad p_i := \sum_{a=1}^{\infty} x_a^i, \quad p'_i := \sum_{a=1}^{\infty} y_a^i,$$

Generating functions for single and double simple Hurwitz numbers (Okounkov, Pandharipande)

$$\tau^{(\gamma,\beta)}(\mathbf{t}) := \tau^{KP(\gamma,\beta)}(0,\mathbf{t}) = \sum_{\lambda} \gamma^{|\lambda|} h_{\lambda}^{-1} e^{\beta\, \mathrm{cont}(\lambda)} s_{\lambda}(\mathbf{t})$$

$$= \sum_{n=0}^{\infty} \gamma^n \sum_{d=0}^{\infty} \frac{\beta^d}{d!} \sum_{\mu, |\mu|=n} H^d_{\exp}(\mu) p_{\mu}(\mathbf{t})$$

$$\tau^{2D(\gamma,\beta)}(\mathbf{t},\mathbf{s}) := \tau^{2DToda(\gamma,\beta)}(0,\mathbf{t},\mathbf{s}) = \sum_{\lambda} \gamma^{|\lambda|} e^{\beta\, \mathrm{cont}(\lambda)} s_{\lambda}(\mathbf{t}) s_{\lambda}(\mathbf{s})$$

$$= \sum_{n=0}^{\infty} \gamma^n \sum_{d=0}^{\infty} \frac{\beta^d}{d!} \sum_{\mu,\nu, |\mu|=\nu|=n} H^d_{\exp}(\mu,\nu) p_{\mu}(\mathbf{t}) p_{\nu}(\mathbf{s})$$

These are therefore **generating functions** for the **simple single and double Hurwitz numbers**.

Weighted Hurwitz numbers: weighted branched coverings

Choose a **weight generating function**

$$G(z) = 1 + \sum_{i=1}^{\infty} g_i z^i$$

For Okounkov-Pandharipande's **simple single and double Hurwitz numbers**: $G(z) = e^z$.

If $G(z)$ is expressible as an **infinite (or finite) product expansion**

$$G(z) := \prod_{i=1}^{\infty}(1 + zc_i), \quad \text{or} \quad G(z) := \prod_{i=1}^{\infty}(1 - zc_i)^{-1}, \quad \mathbf{c} = (c_1, c_2, \dots),$$

the g_i's are the **elementary** or **complete symmetric functions**

$$g_i = e_i(\mathbf{c}), \quad \text{or } g_i = h_i(\mathbf{c}).$$

of the weight determining parameters $\mathbf{c} = (c_1, c_2, \dots)$.

Suppose the **generating function** $G(z)$ and its **dual** $\tilde{G}(z) := \frac{1}{G(-z)}$ can be represented as infinite (or finite) products

$$G(z) = \prod_{i=1}^{\infty}(1 + zc_i), \quad \tilde{G}(z) = \prod_{i=1}^{\infty} \frac{1}{1 - zc_i}.$$

Define the **weight for a branched covering having a pair of branch points with ramification profiles of type (μ, ν), and k additional branch points with ramification profiles $(\mu^{1)}, \dots, \mu^{(k)})$** to be:

$$W_G(\mu^{(1)}, \dots, \mu^{(k)}) := m_\lambda(\mathbf{c}) = \frac{1}{|\mathrm{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \le i_1 < \cdots < i_k} c_{i_{\sigma(1)}}^{\ell^*(\mu^{(1)})} \cdots c_{i_{\sigma(k)}}^{\ell^*(\mu^{(k)})}$$

$$W_{\tilde{G}}(\mu^{(1)}, \dots, \mu^{(k)}) := f_\lambda(\mathbf{c}) = \frac{(-1)^{\ell^*(\lambda)}}{|\mathrm{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \le i_1 \le \cdots \le i_k} c_{i_{\sigma(1)}}^{\ell^*(\mu^{(1)})}, \cdots c_{i_{\sigma(k)}}^{\ell^*(\mu^{(k)})},$$

where the partition λ of length k has **parts $(\lambda_1, \dots, \lambda_k)$ equal to the colengths $(\ell^*(\mu^{(1)}), \dots, \ell^*(\mu^{(k)}))$**, arranged in weakly decreasing order, and $|\mathrm{aut}(\lambda)|$ is the product of the factorials of the multiplicities of the parts of λ.

Definition (Weighted geometrical Hurwitz numbers)

The **weighted geometrical Hurwitz numbers** for *n*-sheeted branched coverings of the Riemann sphere, having a pair of branch points with ramification profiles of type (μ, ν), and *k* additional branch points with ramification profiles $(\mu^{1)}, \dots, \mu^{(k)})$ are defined to be

$$H^d_G(\mu,\nu) := \sum_{k=0}^{\infty} \sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^{k} \ell^*(\mu^{(i)})=d}} W_G(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu)$$

$$H^d_{\tilde{G}}(\mu,\nu) := \sum_{k=0}^{\infty} \sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^{d} \ell^*(\mu^{(i)})=d}} W_{\tilde{G}}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu),$$

where $\sum'$ denotes the sum over all partitions other than the cycle type of the identity element.

Content product formula and τ-function

Choose the following parameters of the hypergeometric τ- function
content product formula

$$r_\lambda^G(\beta) = \prod_{(ij)\in\lambda} G((j-i)\beta) = \prod_{(ij)\in\lambda} r_{j-i}^G(\beta)$$

$$r_j^G(\beta) := G(j\beta) = \frac{\rho_j}{\rho_{j-1}},$$

$$\rho_j = e^{T_j} = \prod_{i=1}^{j} G(i\beta), \quad \rho_0 = 1$$

$$\rho_{-j} = e^{T_{-j}} = \prod_{i=1}^{j} G^{-1}(-i\beta), \quad j = 1, 2, \dots$$

Theorem (Hypergeometric τ-functions as generating function for weighted branched covers)

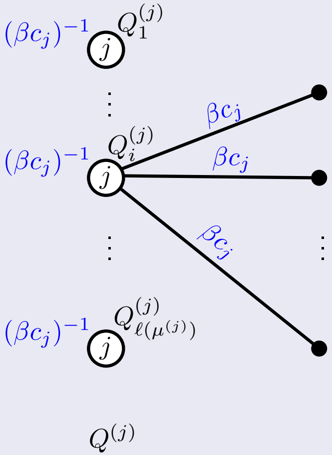
Geometrically,

$$\tau^{\gamma G,\beta}(\mathbf{t},\mathbf{s}) = \sum_{\lambda} \gamma^{|\lambda|} r_\lambda^G(\beta) s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s})$$
$$= \sum_{d=0}^{\infty} \sum_{\substack{\mu,\nu,\\ |\mu|=|\nu|}} \gamma^{|\mu|} H_G^d(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}) \beta^d$$

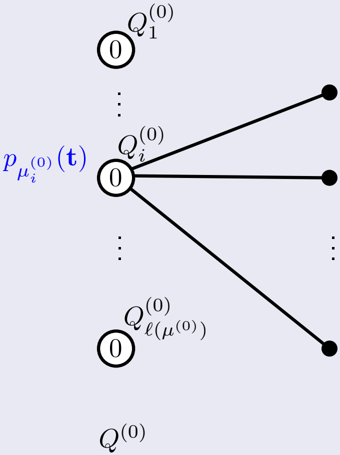
is the generating function for the numbers $H_G^d(\mu,\nu)$ of such weighted n-fold branched coverings of the sphere, with a pair of specified branch points having ramification profiles (μ,ν) and genus given by the **Riemann-Hurwitz formula**

$$2 - 2g = \ell(\mu) + \ell(\nu) - d.$$

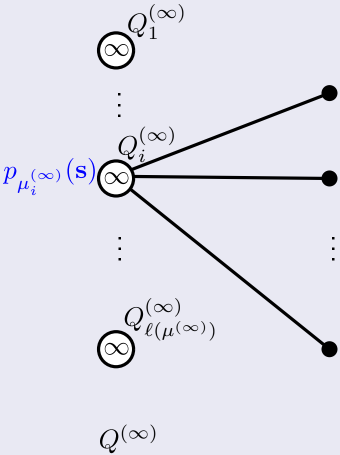
Weighted constellations: Coloured $\{Q^{(j)}_{i=1,\dots,k}\}$ vertices and edges



Weighted constellations: White $\{Q_i^{(0)}\}$ vertices



Weighted constellations: Black $\{Q^{(\infty)_i}\}$ vertices



Paths in the Cayley graph

A ***d*-step path in the Cayley graph of S_n** (generated by all transpositions) is an **ordered sequence**

$$(h,\ (a_1\, b_1)h,\ (a_2\, b_2)(a_1\, b_1)h,\ \dots,\ (a_d\, b_d)\cdots(a_1\, b_1)h)$$

of $d+1$ elements of S_n. If $h \in \mathrm{cyc}(\nu)$ and $g \in \mathrm{cyc}(\mu)$, the path will be referred to as going *from* $\mathrm{cyc}(\nu)$ *to* $\mathrm{cyc}(\mu)$.
If the sequence $b_1, b_2, \dots, b_d$ is either weakly or strictly increasing, then the path is said to be **weakly** (resp. **strictly**) **monotonic**.

Signature of paths

The **signature** λ of a path

$$h\to(a_1 b_1)h\to\cdots\to(a_1 b_1)\cdots(a_{|\lambda|} b_{|\lambda|})h$$

is the partition whose parts equals the number of times a given b_i is repeated.

Path weight

For a partition λ, the product

$$g_\lambda := \prod_{i=1}^{\ell(\lambda)} g_{\lambda_i} = e_\lambda(\mathbf{c}) \quad (\text{ or } \; h_\lambda(\mathbf{c}))$$

is the evaluation, at the weighting parameters $\mathbf{c} = (c_1, c_2, \dots)$, of the **elementary** and **complete** symmetric functions basis elements.

Weighted path enumerative Hurwitz numbers

Let $m^\lambda_{\mu\nu}$ denote the **number of paths** $(a_{|\lambda|} b_{|\lambda|}) \cdots (a_1 b_1) h$ of **signature** λ starting at an element $h \in S_n$ in the conjugacy class $\mathrm{cyc}(\mu)$ with cycle type μ and ending in $\mathrm{cyc}(\nu)$.
Define the (path enumerative) **weighted Hurwitz numbers**

$$F^d_G(\mu,\nu) := \frac{1}{N!} \sum_{\lambda,\ |\lambda|=d} e_\lambda(\mathbf{c}) m^\lambda_{\mu\nu}$$

Theorem (Hypergeometric τ-function as generating function for weighted paths)

It follows that:

$$\tau^{\gamma G,\beta}(\mathbf{t},\mathbf{s}) := \sum_{\lambda} \gamma^{|\lambda|} r_\lambda^G(\beta)^G s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s})$$
$$= \sum_{d=0}^{\infty} \sum_{\substack{\mu,\nu, \\ |\mu|=|\nu|}} \gamma^{|\mu|} \beta^d F_G^d(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}).$$

is the generating function for the numbers $F_G^d(\mu,\nu)$ of weighted d-step paths in the Cayley graph, starting at an element in the conjugacy class of cycle type μ and ending at the conjugacy class of type ν, with weights of all **weakly monotonic paths of type λ** *given by g_λ.*

Corollary (combinatorial-geometrical equivalence)

$$H_G^d(\mu,\nu) = F_G^d(\mu,\nu)$$

Proof, and relation to geometrically weighted Hurwitz numbers: Jucys-Murphy elements

The **Jucys-Murphy elements** $(\mathcal{J}_1, \dots, \mathcal{J}_N)$

$$\mathcal{J}_b := \sum_{a=1}^{b-1} (ab), \quad b = 2, \dots N, \quad \mathcal{J}_1 := 0$$

are a set of **commuting elements of the group algebra C**$[S_n]$

$$\mathcal{J}_a \mathcal{J}_b = \mathcal{J}_b \mathcal{J}_a.$$

Two bases of the center $\mathbf{Z}(\mathbf{C}(S_n))$ of the group algebra

Cycle sums: $C_\mu := \sum_{h \in \mathrm{cyc}(\mu)} h$

Orthogonal idempotents: $F_\lambda := h_\lambda \sum_{\mu, |\mu|=|\lambda|=n} \chi_\lambda(\mu) C_\mu, \quad F_\lambda F_\mu = F_\lambda \delta_{\lambda\mu}$

Theorem (Jucys, Murphy)

If

$$f \in \Lambda_N, \quad f(\mathcal{J}_1, \dots, \mathcal{J}_n) \in \mathbf{Z}(\mathbf{C}[S_N]).$$

and

$$f(\mathcal{J}_1, \dots, \mathcal{J}_N) F_\lambda = f(\{j-i\}) F_\lambda, \quad (ij) \in \lambda.$$

Central element from generating function and Jucys-Murphy

Let

$$G_N(z, \mathbf{x}) = \prod_{a=1}^{N} G(zx_a) \in \Lambda, \qquad \hat{G}_N(z\mathcal{J}) := G_N(z, \mathcal{J}) \in \mathbf{Z}(\mathbf{C}[S_N])$$

Then the eigenvalue is the content product coefficient is $r_\lambda^G(\beta)$

$$G_N(z, \mathcal{J}) F_\lambda = r_\lambda^G(\beta) F_\lambda$$

Weighting factor

The **weighting factor** for paths of signature λ, $|\lambda| = d$ was defined as

$$g_\lambda := \prod_{i=1}^{\ell(\lambda)} g_{\lambda_i}$$

Therefore

$$G(z, \mathcal{J}) C_\mu = \sum_{d=1}^{\infty} Z_\nu F_G^d(\mu, \nu) C_\nu z^d$$

where

$$F_G^d(\mu, \nu) = \frac{1}{N!} \sum_{\lambda, \ |\lambda|=d} e_\lambda(\mathbf{c}) m_{\mu\nu}^\lambda$$

is the **weighted sum** over all such d-step paths, with weight g_λ.

Example 1: Okounkov's simple double Hurwitz numbers

$$G(z) = \exp(z), \quad \exp(z, \mathcal{J}) = \exp(z \sum_{a=1}^{n} \mathcal{J}_a), \quad \exp_j = \frac{1}{j!}$$

$$r_j^{\exp}(\beta) = \exp(j\beta), \quad r_\lambda^{\exp}(\beta) = \prod_{(ij)\in\lambda} \exp(\beta(j-i)),$$

Example 2: Belyi curves: strongly monotone paths

$$G(z) = E(z) := 1 + z, \quad E(z, \mathcal{J}) = \prod_{a=1}^{n} (1 + z\mathcal{J}_a)$$

$$e_1 = 1, \quad e_j = 0 \text{ for } j > 1, \quad r_\lambda^E(\beta) = \prod_{((ij)\in\lambda} (1 + \beta(j-i)),$$

$$T_j^E = \sum_{k=1}^{j} \ln(1 + k\beta), \quad T_{-j}^E = -\sum_{k=1}^{j-1} \ln(1 - k\beta), \quad j > 0.$$

Example 3: Signed Hurwitz numbers: weakly monotone paths

$$G(z) = H(z) := \frac{1}{1-z}, \quad H(z, \mathcal{J}) = \prod_{a=1}^{n} (1 - z\mathcal{J}_a)^{-1}, \quad H_i = 1, \ i \in \mathbf{N}^+$$

$$r_j^H(\beta) = (1 - zj)^{-1}, \quad r_\lambda^H(\beta) = \prod_{(ij)\in\lambda} (1 - \beta(j-i))^{-1},$$

$$T_j^H = -\sum_{i=1}^{j} \ln(1 - i\beta), \quad T_{-j}^E = \sum_{i=1}^{j-1} \ln(1 + i\beta), \quad j > 0.$$

Combinatorially, $H_H^d(\mu, \nu) = F_H^d(\mu, \nu)$ enumerates d-step paths in the Cayley graph of S_n from an element in the conjugacy class of cycle type μ to the class cycle type ν, that are **weakly monotonically increasing** in their second elements.

Signed Hurwitz numbers: weakly monotone paths (cont'd)

Specializing to:

$$t_i := \frac{1}{i} \operatorname{tr} A^N, \; s_i := \frac{1}{i} \operatorname{tr} B^N, \; A, B \in \mathcal{H}^{N \times N}, \; \beta = 1/N$$

Gives the **Itzykson-Zuber-Harish-Chandra integral**

$$\tau^{H, 1/N} = \mathcal{I}_N(A, B) := \int_{U \in U(N)} e^{-\operatorname{tr} UAU^{\dagger} B} d\mu(U)$$
$$= \prod_{k=0}^{N-1} k! \frac{\det(e^{-a_i b_j})_{1 \leq i,j \leq N}}{\Delta(\mathbf{a})\Delta(\mathbf{b})}$$

Signed Hurwitz numbers: weakly monotone paths (cont'd)

The coefficients $H_H^d(\mu,\nu)$ are double Hurwitz numbers that enumerate *n*-sheeted branched coverings of the Riemann sphere with branch points at 0 and ∞ having ramification profile types μ and ν, and an arbitrary number of further branch points, such that the sum of **the complements of their ramification profile lengths** (i.e., the "defect" in the Riemann Hurwitz formula) **is equal to** d. The latter are counted with a sign, which is $(-1)^{n+d}$ times the parity of the number of branch points .

Example 4: Quantum Hurwitz numbers

$$G(z) = E(q,z) := \prod_{k=0}^{\infty}(1+q^k z) = \sum_{k=0}^{\infty} E_k(q) z^k,$$

$$= e^{-\operatorname{Li}_2(q,-z)}, \quad \operatorname{Li}_2(q,z) := \sum_{k=1}^{\infty} \frac{z^k}{k(1-q^k)} \text{ (quantum dilogarithm)}$$

$$E_i(q) := \prod_{j=0}^{i} \frac{q^j}{1-q^j},$$

$$E(q,\mathcal{J}) = \prod_{a=1}^{n}\prod_{i=0}^{\infty}(1+q^i z \mathcal{J}_a),$$

$$r_j^{E(q)}(\beta) = \prod_{k=0}^{\infty}(1+q^k \beta j), \quad r_\lambda^{E(q)}(\beta) = \prod_{k=0}^{\infty}\prod_{(ij)\in\lambda}(1+q^k\beta(j-i)),$$

$$T_j^{E(q)} = -\sum_{i=1}^{j} \operatorname{Li}_2(q,-zi).$$

Symmetrized monotone monomial sums

Using the sums:

$$\sum_{\sigma\in S_k}\sum_{0\leq i_1<\cdots<i_k}^{\infty} x_{\sigma(1)}^{i_1}\cdots x_{\sigma(k)}^{i_k}$$

$$=\sum_{\sigma\in S_k}\frac{x_{\sigma(1)}^{k-1}x_{\sigma(2)}^{k-2}\cdots x_{\sigma(k-1)}}{(1-x_{\sigma(1)})(1-x_{\sigma(1)}x_{\sigma(2)})\cdots(1-x_{\sigma(1)}\cdots x_{\sigma(k)})}$$

$$\sum_{\sigma\in S_k}\sum_{1\leq i_1<\cdots<i_k}^{\infty} x_{\sigma(1)}^{i_1}\cdots x_{\sigma(k)}^{i_k}$$

$$=\sum_{\sigma\in S_k}\frac{x_{\sigma(1)}^{k}x_{\sigma(2)}^{k-1}\cdots x_{\sigma(k)}}{(1-x_{\sigma(1)})(1-x_{\sigma(1)}x_{\sigma(2)})\cdots(1-x_{\sigma(1)}\cdots x_{\sigma(k)})}$$

Theorem (Quantum Hurwitz numbers (cont'd))

$$\tau^{\gamma E(q,)\beta}(\mathbf{t},\mathbf{s}) = \sum_{k=0}^{\infty} \beta^k \sum_{\substack{\mu,\nu \\ |\mu|=|\nu|}} H^d_{E(q)}(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}), \quad \textit{where}$$

$$H^d_{E(q)}(\mu,\nu) := \sum_{d=0}^{\infty} \sideset{}{'}\sum_{\substack{\mu^{(1)},\dots\mu^{(d)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}} W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu),$$

$$\textit{with} \quad W_{E(q)}(\mu^{(1)},\dots,\mu^{(k)}) := \frac{1}{k!} \sum_{\sigma\in S_k} \sum_{0\le i_1<\dots<i_k}^{\infty} q^{i_1\ell^*(\mu^{(\sigma(1))})} \cdots q^{i_k\ell^*(\mu^{(\sigma(k))})}$$

$$= \frac{1}{k!} \sum_{\sigma\in S_k} \frac{q^{(k-1)\ell^*(\mu^{(\sigma(1))})} \cdots q^{\ell^*(\mu^{(\sigma(k-1))})}}{(1-q^{\ell^*(\mu^{(\sigma(1))})})\cdots(1-q^{\ell^*(\mu^{(\sigma(1))})}\cdots q^{\ell^*(\mu^{(\sigma(k))})})}$$

are the **weighted (quantum) Hurwitz numbers** *that count the number of branched coverings with genus g given by the* **Riemann-Hurwitz formula***:* $2-2g = \ell(\lambda)+\ell(\mu)-k$*. and sum of colengths d.*

Corollary (Quantum Hurwitz numbers and quantum paths)

The **weighted sum** *over d-step paths in the Cayley graph from an element of the conjugacy class μ to one in the class ν*

$$F^d_{E(q)}(\mu,\nu) := \frac{1}{n!} \sum_{\lambda,\ |\lambda|=d} E_\lambda(q) m^\lambda_{\mu\nu}, \quad E_\lambda(q) = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} \frac{q^j}{1-q^j}$$

is equal to the **weighted Hurwitz number**

$$F^d_{E(q)}(\mu,\nu) = H^d_{E(q)}(\mu,\nu)$$

counting **weighted n-sheeted branched coverings** *of* $\mathbf{P}^1$ *with a pair of branched points of ramification profiles μ and ν, and any number of further branch points, and genus determined by the Riemann-Hurwitz formula*

$$2-2g = \ell(\mu) + \ell(\nu) - d$$

and these are generated by the τ function $\tau^{E(q,z)}(\mathbf{t},\mathbf{s})$.

Bosonic gases and Planck's distribution law

A slight modification consists of replacing the generating function $E(q,z)$ by

$$E'(q,z) := \prod_{k=1}^{\infty}(1+q^k z).$$

The effect of this is simply to replace the weighting factors

$$\frac{1}{1-q^{\ell^*(\mu)}} \quad \text{by} \quad \frac{1}{q^{-\ell^*(\mu)}-1}.$$

If we identify

$$q := e^{-\beta\hbar\omega_0}, \quad \beta = 1/k_B T,$$

where ω_0 is the lowest frequency excitation in a **gas of identical bosonic particles** and assume the energy spectrum of the particles consists of integer multiples of $\hbar\omega_0$

$$\epsilon_j = j\hbar\omega_0,$$

Expectation values of Hurwitz numbers

The relative probability of occupying the energy level ϵ_k is

$$\frac{q^k}{1-q^k} = \frac{1}{e^{\beta\epsilon_k}-1},$$

the **energy distribution of a bosonic gas**.

We may associate the branch points to the states of the gas and view the Hurwitz numbers $H(\mu^{(1)}, \dots \mu^{(l)})$ as **random variables**, with the state energies proportional to the sums over the colengths

$$\epsilon(\mu^{(i)}) := \epsilon_{\ell^*(\mu^{(i)})} = \ell^*(\mu^{(i)})\beta\hbar\omega_0,$$

and weight

$$\frac{q^{\ell^*(\mu^{(i)})}}{1-q^{\ell^*(\mu^{(i)})}} = \frac{1}{e^{\beta\epsilon_{\ell^*(\mu^{(i)})}}-1}$$

Expectation values of quantum Hurwitz numbers

the normalized quantum Hurwitz numbers are expectation values

$$\bar{H}^d_{E'(q)}(\mu,\nu) := \frac{1}{\mathbf{Z}^d_{E'(q)}} \sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}} W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu)$$

where $W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}) = \frac{1}{k!}\sum_{\sigma\in S_k} W(\mu^{(\sigma(1)}) \cdots W(\mu^{(\sigma(1)},\cdots,\mu^{(\sigma(k)})$

$$W(\mu^{(1)},\dots,\mu^{(k)}) := \frac{1}{e^{\beta\sum_{i=1}^k \epsilon(\mu^{(i)})}-1},$$

$$\mathbf{Z}^d_{E'(q)} := \sum_{k=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots\mu^{(k)} \\ \sum_{i=1}^k \ell^*(\mu^{(i)})=d}} W_{E'(q)}(\mu^{(1)},\dots,\mu^{(k)}).$$

is the **canonical partition function** for total energy $d\hbar\omega$.

Fermionic representation of hypergeometric 2D Toda τ-functions

The fermionic creation and annihiliation operators $\{\psi_i, \psi_i^\dagger\}_{i\in\mathbf{Z}}$ satisfy the usual **anticommutation relations** and **vacuum state** $|0\rangle$ vanishing conditions

$$[\psi_i, \psi_j^\dagger]_+ = \delta_{ij} \quad \psi_i|0\rangle = 0, \quad \text{for } i < 0, \quad \psi_i^\dagger|0\rangle = 0, \quad \text{for } i \geq 0.$$

The **shift flow** abelian subgroups $\Gamma_+ = \{\gamma_+(\mathbf{t}\},\ \Gamma_- = \{\gamma_-(\mathbf{s})\}$ are defined in terms of the **current components** $J\{_i\}_{i\in\mathbf{Z}}$ as

$$\hat{\gamma}_+(\mathbf{t}) = e^{\sum_{i=1}^\infty t_i J_i}, \quad \hat{\gamma}_-(\mathbf{s}) = e^{\sum_{i=1}^\infty s_i J_{-i}}, \quad J_i = \sum_{k\in\mathbf{Z}} \psi_k \psi_{k+i}^\dagger, \quad i \in \mathbf{Z}.$$

Choose the parameters of the hypergeometric τ- function as:

$$\rho_j = e^{T_i} = \prod_{i=1}^{j} G(i\beta), \quad \rho_{-j} = e^{T_{-i}} = \prod_{i=0}^{j-1} (G(-i\beta))^{-1}, \quad j = 1, 2, \dots.$$

Fermionic expressions for the τ-function and Baker function

The **τ-function** is expressed fermionically as

$$\tau_\rho(N, \mathbf{t}, \mathbf{s}) := \langle N | \hat{\gamma}_+(\mathbf{t}) \hat{C}_\rho \hat{\gamma}_-(\mathbf{s}) | N \rangle$$

where

$$\hat{C}_\rho := e^{\sum_{i \in \mathbf{Z}} (T_i + i \ln(\gamma)) :\psi_i \psi_i^\dagger:}$$

The **Baker function and dual Baker function** are

$$\Psi_\rho(z, \mathbf{t}, \mathbf{s}) = \frac{\langle 0 | \psi_0^\dagger \hat{\gamma}_+(\mathbf{t}) \psi(z) \hat{C}_\rho \hat{\gamma}_-(\mathbf{s}) | 0 \rangle}{\tau_\rho(N, \mathbf{t}, \mathbf{s})},$$

$$\Psi^*_\rho(z, \mathbf{t}, \mathbf{s}) = \frac{\langle 0 | \psi_{-1} \hat{\gamma}_+(\mathbf{t}) \psi^\dagger(z) \hat{C}_\rho \hat{\gamma}_-(\mathbf{s}) | 0 \rangle}{\tau_\rho(N, \mathbf{t}, \mathbf{s})}.$$

$$\psi(z) := \sum_{i \in \mathbf{Z}} \psi_i z^i, \qquad \psi^\dagger(z) := \sum_{i \in \mathbf{Z}} \psi_i^\dagger z^{-i-1},$$

Fermionic representation of adapted bases

Adapted basis

$$\Psi_k^+(x) := \begin{cases} \gamma\langle 0|\psi^*(1/x)\hat{C}_\rho\hat{\gamma}_-(\beta^{-1}\mathbf{s})\psi_{k-1}|0\rangle & \text{if } k \geq 1, \\ \gamma\langle 0|\psi_{k-1}\hat{\gamma}_-^{-1}(\beta^{-1}\mathbf{s})\ \hat{C}_\rho^{-1}\psi^*(1/x)\hat{C}_\rho\hat{\gamma}_-(\beta^{-1}\mathbf{s})|0\rangle & \text{if } k \leq 0, \end{cases}$$

$$= \gamma\sum_{j=0}^{\infty}\rho_{j+k-1}h_j(\beta^{-1}\mathbf{s})x^{j+k}$$

Dual adapted basis

$$\Psi_k^-(x) := \begin{cases} \langle 0|\psi(1/x)\hat{C}_\rho\hat{\gamma}_-(\beta^{-1}\mathbf{s})\psi^*_{-k}|0\rangle & \text{if } k \geq 1 \\ \langle 0|\psi^*_{-k}\hat{\gamma}_-^{-1}(\beta^{-1}\mathbf{s})\ \hat{C}_\rho^{-1}\psi(1/x)\hat{C}_\rho\hat{\gamma}_-(\beta^{-1}\mathbf{s})|0\rangle & \text{if } k \leq 0. \end{cases}$$

$$= \sum_{j=0}^{\infty}\rho^{-1}_{-j-k}h_j(-\beta^{-1}\mathbf{s})x^{j+k}$$

Recursion operator

Define the **recursion operators**

$$R_{\pm}(x) := \gamma x G(\pm \beta D),$$

where

$$D := x\frac{d}{dx}$$

is the **Euler operator**.
Then

$$\Psi^{+}_{k\pm1}(x) := R_{+}^{\pm1}\Psi^{+}_{k}(x),$$
$$\Psi^{-}_{k\pm1}(x) := R_{-}^{\pm1}\Psi^{-}_{k}(x).$$

Theorem (Quantum spectral curve at $\mathbf{t} = 0$)

The function $\Psi_0^+(x)$ *satisfies*

$$(\beta D - S(R_+))\,\Psi_0^+(x) = 0,$$

and $\Psi_0^-(x)$ *satisfies the dual equation*

$$(\beta D + S(R_-))\,\Psi_0^-(x) = 0.$$

where

$$S(x) := \sum_{k=1}^{\infty} k s_k x^k.$$

Classical spectral curve at $\mathbf{t} = 0$

$$\begin{cases} x = \frac{z}{\gamma G(S(z))} \\ y = \frac{S(z)}{z}\gamma G(S(z)) \end{cases}$$

Pair correlator

The **pair correlator** is

$$K(x,x') := \frac{1}{x-x'}\,\tau^{(G,\beta,\gamma)}([x]-[x'],\beta^{-1}\mathbf{s}) = \frac{1}{xx'}\langle 0|\psi(1/x')\psi^*(1/x)\hat{C}_\rho\hat{\gamma}_-(\beta^{-1}\mathbf{s})|0\rangle. \qquad (3.1)$$

Finite degree

In the following, we assume that only a finite number of variables s_k are nonzero, so $S(x)$ is a polynomial

$$S(x) = \sum_{k=1}^{L} k s_k x^k$$

of degree L. We also assume that only the first M parameters $\{c_i\}$ are nonzero, so the weight generating function $G(z)$ is also a polynomial, of degree M

Christoffel-Darboux-type reduction

Define the operators

$$\Delta_\pm(x) := S(x) \pm \beta D$$
$$V_\pm(x) := G(\Delta_\pm(x))$$

and the **Christoffel Darboux**-type kernel generating function

$$A(r,t) := (r\,V_-(t) - t\,V_+(r))\left(\frac{1}{r-t}\right) = \sum_{i=0}^{LM-1}\sum_{j=0}^{LM-1} \mathbf{A}_{ij} r^i t^j.$$

Theorem

The following **Christoffel-Darboux** *type relation expresses* $K(x,x')$ *as a finite rank sum in terms of the adapted bases*

$$K(x,x') = \frac{1}{x-x'} \sum_{i=0}^{LM-1}\sum_{j=0}^{LM-1} \mathbf{A}_{ij} r^i t^j \Psi_i^+(x)\Psi_j^-(x').$$

Definition (Current correlator generating function)

$$F_n(\mathbf{s}; x_1, \dots, x_n) := \sum_{\mu,\nu,\,\ell(\mu)=n} \sum_d \gamma^{|\mu|} \beta^{d-\ell(\nu)} H_G^d(\mu,\nu)\, \tilde{z}_\mu m_\mu(x_1,\dots,x_n) p_\nu(\mathbf{s})$$

$$\tilde{F}_n(\mathbf{s}; x_1, \dots, x_n) := \sum_{\mu,\nu,\,\ell(\mu)=n} \sum_d \gamma^{|\mu|} \beta^{d-\ell(\nu)} \tilde{H}_G^d(\mu,\nu)\, \tilde{z}_\mu m_\mu(x_1,\dots,x_n) p_\nu(\mathbf{s})$$

$$\tilde{F}_{g,n}(\mathbf{s}; x_1, \dots, x_n) := \sum_{\mu,\nu,\,\ell(\mu)=n} \gamma^{|\mu|} \tilde{H}_G^d(\mu,\nu)\, \tilde{z}_\mu m_\mu(x_1,\dots,x_n) p_\nu(\mathbf{s}) ,$$

Current correlators as generating functions for weighted Hurwitz numbers

The **fermionic current correlator** is a **generating function** for (connected) weighted Hurwitz numbers $\tilde{H}^d(\mu, \nu)$ at fixed $\ell(\mu) = n$

$$W_n(\mathbf{s}; x_1, \dots, x_n) := = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_n(\mathbf{s}; x_1, \dots, x_n),$$

$$:= \frac{1}{\prod_{i=1}^n x_i} \langle 0 | \prod_{i=1}^n J_+(x_i) \hat{C}_\rho \hat{\gamma}_-(\beta^{-1}\mathbf{s}) | 0 \rangle,$$

where $J_+(x)$ is the positive part of the current generator

$$J_+(x) := \sum_{i=1}^{\infty} J_i x^i$$

Definition

$$\tilde{\omega}_{g,n}(z_1,\dots,z_n) := \tilde{W}_{g,n}(X(z_1),\dots,X(z_n))\, X'(z_1)\dots X'(z_n)dz_1\dots dz_n + \delta_{g,0}\delta_{n,2}\, \frac{X'(z_1)X'(z_2)\, dz_1 dz_2}{(X(z_1)-X(z_2))^2}$$

Theorem

When $2g-2+n>0$, $\tilde{\omega}_{g,n}(z_1,\dots,z_n)$ has poles only at branch points, and no poles at $z_i=\infty$.

Theorem (Topological recursion)

$\tilde{\omega}_{g,n}$ satisfy the **topological recursion** *relations. If all branchpoints a are simple, with local Galois involution $z \mapsto \sigma_a(z)$, we have*

$$\tilde{\omega}_{g,n}(z_1,\dots,z_n) = -\sum_a \operatorname*{res}_{z=a} \left[\frac{dz_1}{z-z_1} - \frac{dz_1}{\sigma_a(z)-z_1}\right] \frac{\mathcal{W}_{g,n}(z,\sigma_a(z),z_2,\dots,z_n)}{2(Y(z)-Y(\sigma_a(z)))\,dX(z)}$$

$$\text{where}\quad \mathcal{W}_{g,n}(z,z',z_2,\dots,z_n) = \tilde{\omega}_{g-1,n+1}(z,z',z_2,\dots,z_n) + \sum_{g_1+g_2=g,\ I_1\uplus I_2=\{z_2,\dots,z_n\}}^{\prime} \tilde{\omega}_{g_1,1+|I_1|}(z,I_1)\tilde{\omega}_{g_2,1+|I_2|}(z',I_2)$$

References

- A. Alexandrov, G. Chapuy, B. Eynard and J. Harnad, "Weighted Hurwitz numbers and topological recursion: summary overview", (CRM preprint, 2016, arXiv:1610.09408).
- Vincent Bouchard and Bertrand Eynard, "Think globally, compute locally ", JHEP 1302 (2013) 143 (arXiv:1211.2302).
- M. Guay-Paquet and J. Harnad, "2D Toda τ-functions as combinatorial generating functions", arxiv:1405.6303. *Lett. Math. Phys.* **105,** 827-852 (2015).
- M. Guay-Paquet and J. Harnad, "Generating functions for weighted Hurwitz numbers", arxiv:1408.6766. *J Math. Phys,* in press (2017).
- J. Harnad and A. Yu. Orlov, "Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths", *Commun. Math. Phys.* **338**, 267-284 (2015).
- J. Harnad, "Multispecies weighted Hurwitz numbers" *Sigma* **11** 097 (2015).
- J. Harnad, "Quantum Hurwitz numbers and Macdonald polynomials", *J. Math. Phys.* **57,** 113505 (2016).
- J. Harnad, 'Weighted Hurwitz numbers and hypergeometric τ-functions: an overview", *AMS Proceedings of Symposia in Pure Mathematics* **93**, 289-333 (2016).
- A. Okounkov, "Toda equations for Hurwitz numbers", *Math. Res. Lett.* **7**, 447-453 (2000).