Critical Boltzmann Ising triangulations of the half plane

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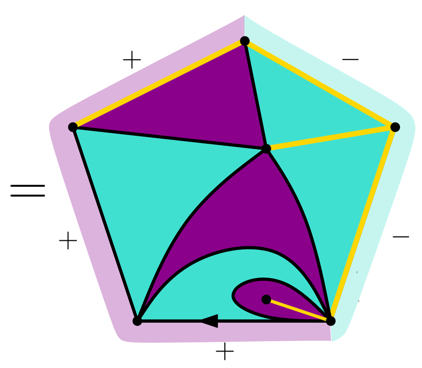
Triangulations of a polygon

A *triangulation of the p-gon of volume n* is a gluing of n triangles along their edges such that the resulting surface is a topological disk with p edges on its boundary.
The triangulations considered here are *rooted* and *bicolored*. That is, they come with one marked oriented edge on the boundary (the root edge) and each of their *faces* is bicolored (alternatively, comes with a spin + or −).

$\mathfrak{b} =$ 

$$\mathrm{Vol}(\mathfrak{b}) = 7$$
$$m(\mathfrak{b}) = 5$$
$$\mathbb{P}(\mathfrak{B}_{3,2} = \mathfrak{b}) = \frac{t^7 \nu^5}{z_{3,2}(t,\nu)}$$

Figure 1: An example of bicolored rooted triangulation of the pentagon with a boundary condition $+^3-^2$. The monochromatic edges are highlighted.

Boltzmann Ising triangulations

For all $p, q \geq 0$ such that $p + q \geq 1$, define

$$\mathbb{B}_{p,q} = \left\{ \begin{array}{l} \text{bicolored rooted triangulations of} \\ \text{the } (p+q)\text{-gon with a Dobrushin} \\ \text{boundary condition of type } +^p-^q. \end{array} \right\}$$

For $\mathfrak{b} \in \mathbb{B}_{p,q}$, denote by $\mathrm{Vol}(\mathfrak{b})$ its volume and by $m(\mathfrak{b})$ the number of monochromatic edges.

A *Boltzmann Ising triangulation of the (p,q)-gon* is a random variable $\mathfrak{B}_{p,q}$ on $\mathbb{B}_{p,q}$ of law

$$\mathbb{P}(\mathfrak{B}_{p,q} = \mathfrak{b}) = \frac{t^{\mathrm{Vol}(\mathfrak{b})} \nu^{m(\mathfrak{b})}}{z_{p,q}(t,\nu)},$$

where $t, \nu > 0$ and

$$z_{p,q}(t,\nu) = \sum_{\mathfrak{b} \in \mathbb{B}_{p,q}} t^{\mathrm{Vol}(\mathfrak{b})} \nu^{m(\mathfrak{b})}.$$

We define the generating function

$$Z(u,v;t,\nu) = \sum_{p,q \geq 0} z_{p,q}(t,\nu)\, u^p v^q$$

where by convention $z_{0,0} = 1$.

Localization of the critical point

Theorem (Bernardi, Bousquet-Mélou 2011) [1]

$z_{1,0}(t,\nu)$ is an *algebraic function* which has an explicit rational parametrization. Moreover,

$$[t^n] z_{1,0}(t,\nu) \underset{n \to \infty}{\sim} c(\nu) \tau(\nu)^{-n} n^{-\alpha(\nu)}$$

where $c(\nu)$, $\tau(\nu)$ are continuous in ν and

$$\alpha(\nu) = \begin{cases} 5/2 & \nu \neq \nu_c \\ 7/3 & \nu = \nu_c \end{cases}$$

with $\nu_c = 1 + 2\sqrt{7}$ and $t_c = \tau(\nu_c) = \frac{\sqrt{5}\sqrt{35-11\sqrt{7}}}{28 \cdot 6^{3/2}}$.
We have $z_{1,0}(\tau(\nu), \nu) < \infty$ for all $\nu \geq 0$.

Asymptotic enumeration of critical Boltzmann Ising triangulations

In this work we concentrate on the critical phase where $(t,\nu) = (t_c, \nu_c)$. Its definition is given by the result on the left.

Theorem 1

$(u,v) \mapsto Z(u,v;t_c,\nu_c)$ is an *algebraic function* which has an explicit rational parametrization. Moreover,

$$z_{p,q}(t_c,\nu_c) \underset{p \to \infty}{\sim} C_q\, u_c^{-p} p^{-7/3}$$
$$C_q \underset{q \to \infty}{\sim} D\, u_c^{-q} q^{-4/3}$$

where $u_c = \frac{6}{5}(7 + \sqrt{7}) t_c \approx 0.152$.

Main result

For any rooted bicolored triangulation $\mathfrak{b} \in \mathbb{B} := \cup_{p,q \geq 0} \mathbb{B}_{p,q}$, denote by $B_r(\mathfrak{b})$ the metric ball of radius r around the root vertex in $\mathfrak{b}$. The *local (Benjamini-Schramm) distance* on $\mathbb{B}$ is defined by

$$d_{\mathrm{loc}}(\mathfrak{b}, \mathfrak{b}') = 2^{-R} \quad \text{with} \quad R = \sup\{ r \geq 0 \mid B_r(\mathfrak{b}) = B_r(\mathfrak{b}') \}$$

It is well known that the Cauchy completion $\overline{\mathbb{B}}$ of $\mathbb{B}$ with respect to $d_{\mathtt{loc}}$ is a Polish space.

Theorem 2

There exist $\mathfrak{B}_{\infty,q} \in \overline{\mathbb{B}}$ and $\mathfrak{B}_{\infty,\infty}\overline{\mathbb{B}}$ such that we have the following convergence in distribution under $d_{\mathtt{loc}}$

$$\mathfrak{B}_{p,q} \xrightarrow[p \to \infty]{} \mathfrak{B}_{\infty,q} \quad \text{and} \quad \mathfrak{B}_{\infty,q} \xrightarrow[q \to \infty]{} \mathfrak{B}_{\infty,\infty}$$

Moreover, $\mathfrak{B}_{\infty,q}$ and $\mathfrak{B}_{\infty,\infty}$ are triangulations of the half-plane, i.e. they are one-ended and have an infinite boundary.

Theorem 3

Let L_q be the perimeter of the largest cluster of − triangles attached to the − boundary of $\mathfrak{B}_{\infty,q}$. Then we have the convergence in distribution

$$\frac{L_q}{q} \xrightarrow[q \to \infty]{} L_\infty$$

where the law of L_∞ is given by

$$\mathbb{P}(L_\infty > x) = (1 + \mu x)^{-\gamma}$$

with $\mu = \frac{1}{4\sqrt{7}}$ and $\gamma = \frac{4}{3}$.

In particular, this suggests that the Hausdorff dimension of the Ising boundary under quantum volume measure should be 1.

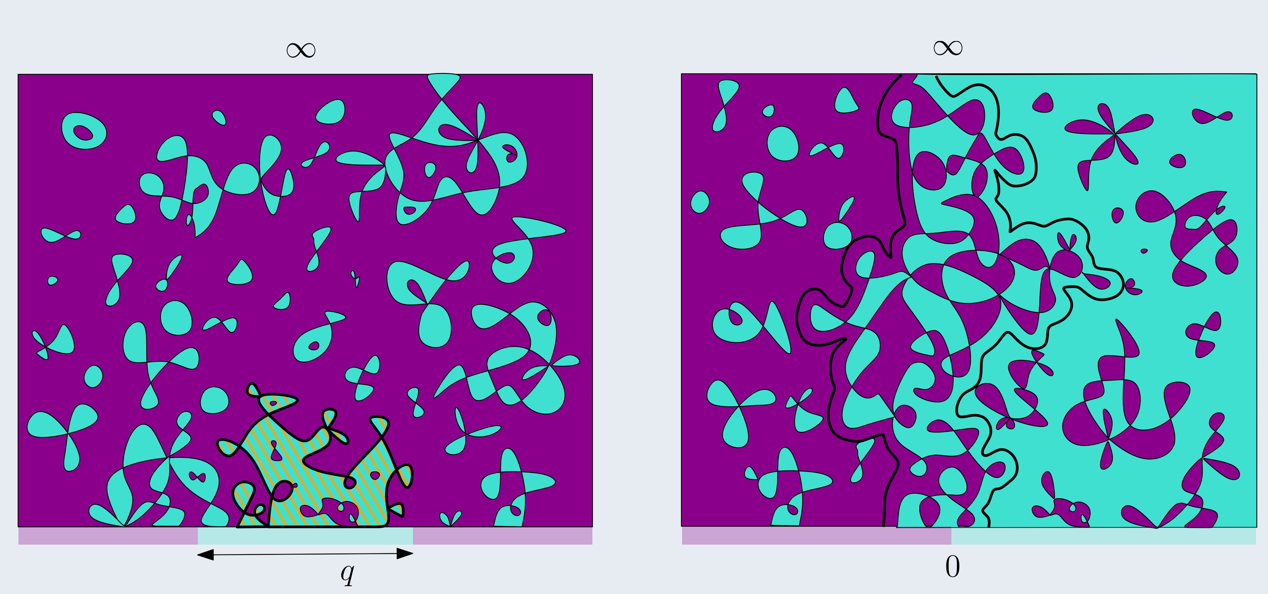


Figure 2: An artistic view of the cluster structure of the infinite bicolored triangulations $\mathfrak{B}_{\infty,q}$ and $\mathfrak{B}_{\infty,\infty}$. Almost surely, all − clusters in $\mathfrak{B}_{\infty,q}$ are finite, and there is exactly one infinite + cluster. $\mathfrak{B}_{\infty,\infty}$ is obtained by gluing $\mathfrak{B}_{\infty,0}$ and $\mathfrak{B}_{0,\infty}$ to both sides of a "ribbon". The three pieces are mutually independent. The largest − cluster mentioned in Theorem 3 is highlighted.

Idea of proofs

Theorem 1: Write Tutte's equation associated to the simple peeling process which explores the Ising interface imposed by the boundary condition. Solve it using a generalization of the quadratic method similar to the one used in [1]. The asymptotics are obtained with the standard machinery of analytic combinatorics [3].
Theorem 2: Show that the outcome of each step in any peeling process on $\mathfrak{B}_{p,q}$ converges in distribution. Choose a peeling algorithm such that for all $r \geq 0$, the balls $B_r(\mathfrak{B}_{\infty,q})$ and $B_r(\mathfrak{B}_{\infty,\infty})$ can be reconstructed from finitely many peeling steps with high probability. The proof is centered around the study of Markov chains which track the evolution of + and − boundary length during the peeling process. The step distributions of these Markov chains have a positive drift and a polynomial heavy-tail, which allows us to use ideas from [2].
Theorem 3: One peeling process used in the proof of Theorem 2 explores the boundary of the − clusters attached to the − boundary of $\mathfrak{B}_{\infty,q}$. The perimeters of the clusters are encoded as the sizes of jumps of a Markov chain. In particular, L_q is approximately the time of the unique big jump of this Markov chain.

References

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