# Does Eulerian percolation on the square lattice percolate?

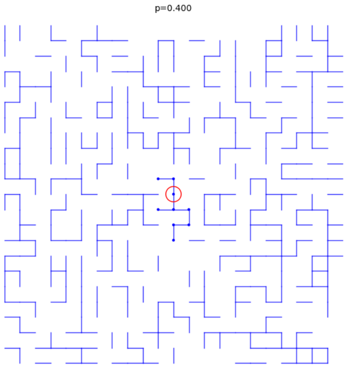
Irène Marcovici

Joint work with **Régine Marchand** and **Olivier Garet**

Institut Élie Cartan de Lorraine, Univ. de Lorraine, Nancy

IHP, January 16th, 2017

# Classical Bernoulli bond percolation



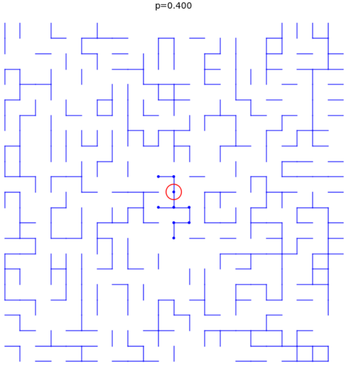

$\omega \in \{0,1\}^{\mathbb{E}_2}$

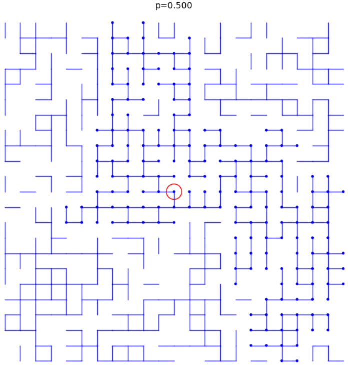
$(\omega_e)_{e \in \mathbb{E}_2}$ i.i.d. with law Ber($p$)

$\mathbb{E}_2$ = set of edges of $\mathbb{Z}^2$

$\omega_e = 1$ if edge $e$ is colored in blue

# Classical Bernoulli bond percolation



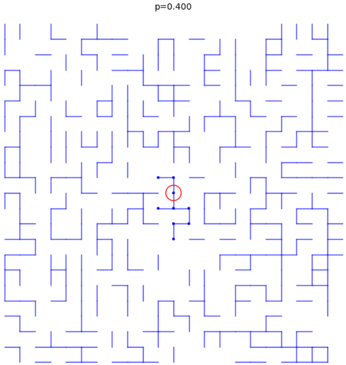

$\omega \in \{0,1\}^{\mathbb{E}_2}$

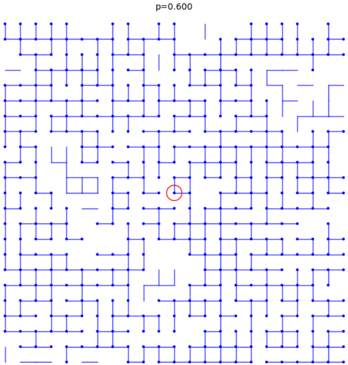
$(\omega_e)_{e \in \mathbb{E}_2}$ i.i.d. with law Ber$(p)$

$\mathbb{E}_2$ = set of edges of $\mathbb{Z}^2$

$\omega_e = 1$ if edge $e$ is colored in blue

# Classical Bernoulli bond percolation



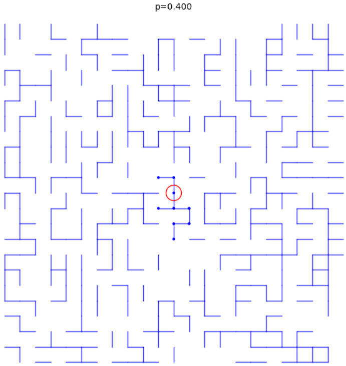

$\omega \in \{0,1\}^{\mathbb{E}_2}$

$(\omega_e)_{e\in\mathbb{E}_2}$ i.i.d. with law $\mathrm{Ber}(p)$

$\mathbb{E}_2$ = set of edges of $\mathbb{Z}^2$

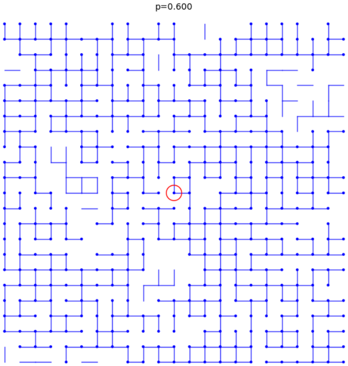
$\omega_e = 1$ if edge $e$ is colored in blue

# Classical Bernoulli bond percolation




$p < 0.5$

No infinite connected component




$p > 0.5$

Infinite connected component

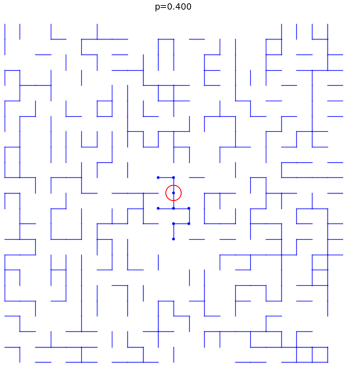
$\omega \in \{0,1\}^{\mathbb{E}_2}$

$(\omega_e)_{e \in \mathbb{E}_2}$ i.i.d. with law Ber($p$)

$\mathbb{E}_2$ = set of edges of $\mathbb{Z}^2$

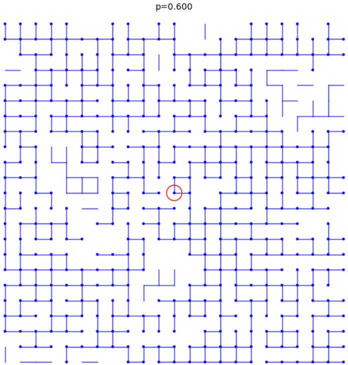
$\omega_e = 1$ if edge $e$ is colored in blue

# Classical Bernoulli bond percolation




$p \leq 0.5$

No infinite connected component




$p > 0.5$

Infinite connected component

$\omega \in \{0,1\}^{\mathbb{E}_2}$

$(\omega_e)_{e \in \mathbb{E}_2}$ i.i.d. with law Ber($p$)

$\mathbb{E}_2$ = set of edges of $\mathbb{Z}^2$

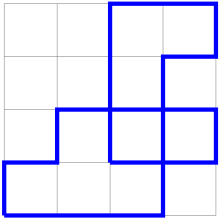
$\omega_e = 1$ if edge $e$ is colored in blue

## Eulerian percolation (= even percolation)

Bernoulli bond percolation with parameter $p$, conditioned on the fact that every site has an even degree

# Eulerian percolation (= even percolation)

Bernoulli bond percolation with parameter $p$, conditioned on the fact that every site has an even degree

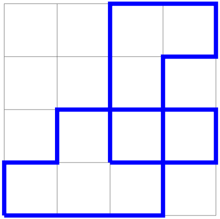


Probability $\frac{1}{Z_p} \, p^{N_b}(1-p)^{N_g}$

$N_b$ =number of blue edges
$N_g$ =number of grey edges

## Eulerian percolation (= even percolation)

Bernoulli bond percolation with parameter $p$, conditioned on the fact that every site has an even degree



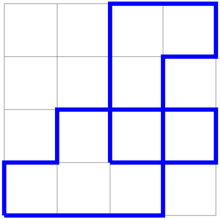
Probability $$\frac{1}{Z_p}\, p^{N_b}(1-p)^{N_g}$$

$N_b$ =number of blue edges
$N_g$ =number of grey edges

How to define the even percolation measure on the whole $\mathbb{Z}^2$?

## Eulerian percolation (= even percolation)

Bernoulli bond percolation with parameter $p$, conditioned on the fact that every site has an even degree



Probability $$\frac{1}{Z_p} \, p^{N_b}(1-p)^{N_g}$$

$N_b$ =number of blue edges
$N_g$ =number of grey edges

How to define the even percolation measure on the whole $\mathbb{Z}^2$?
What are the connectivity properties of the random (blue) subgraph obtained?

## Definition of the even percolation measure on $\mathbb{Z}^2$

Degree of vertex $x$ in configuration $\omega$: $d_\omega(x) = \sum_{e \ni x} \omega_e$

We want to condition the Bernoulli bond percolation to the event:

$$\Omega_{EP} = \{\omega \in \{0,1\}^{\mathbb{E}_2}; \forall x \in \mathbb{Z}^2, d_\omega(x) \equiv 0[2]\}$$

## Definition of the even percolation measure on $\mathbb{Z}^2$

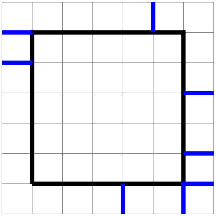
Degree of vertex $x$ in configuration $\omega$: $d_\omega(x) = \sum_{e \ni x} \omega_e$

We want to condition the Bernoulli bond percolation to the event:

$$\Omega_{EP} = \{\omega \in \{0,1\}^{\mathbb{E}_2}; \forall x \in \mathbb{Z}^2, d_\omega(x) \equiv 0[2]\}$$

**Gibbs measures** formalism:

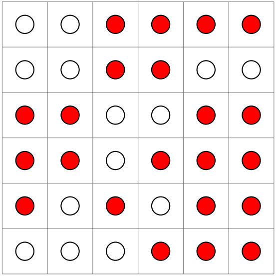
$$\mu^p_{\Lambda,\eta}(\omega) = \frac{1}{Z} \mathbf{1}_{\eta_{\Lambda^c}\omega_\Lambda \in \Omega_{EP}} \; p^{N_b(\omega_\Lambda)} \; (1-p)^{N_g(\omega_\Lambda)}$$
$$= \frac{1}{Z'} \mathbf{1}_{\eta_{\Lambda^c}\omega_\Lambda \in \Omega_{EP}} \left(\frac{p}{1-p}\right)^{N_b(\omega_\Lambda)}$$



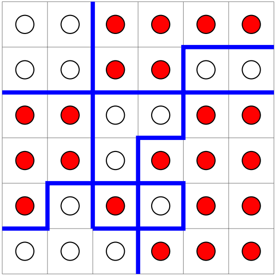
Finite box $\Lambda$

Configuration $\eta \in \Omega_{EP}$ outside $\Lambda$

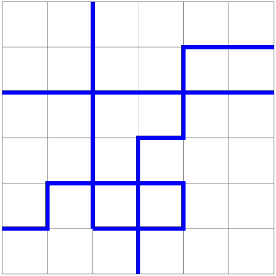
# Colorings and contours



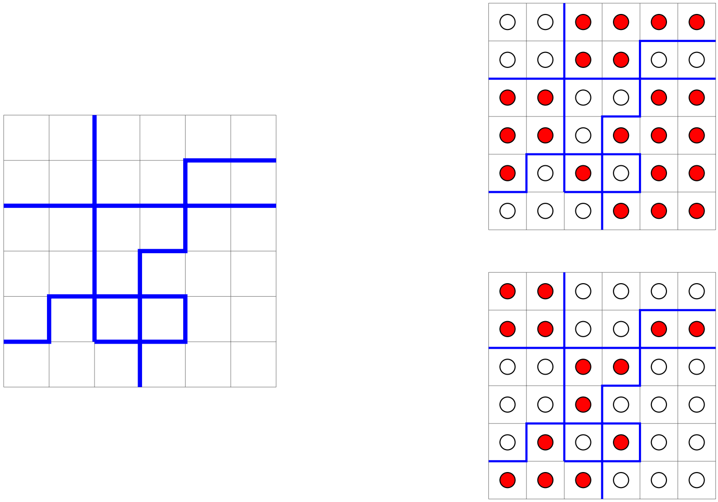
# Colorings and contours



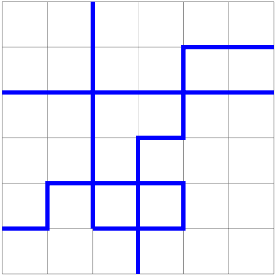
# Colorings and contours

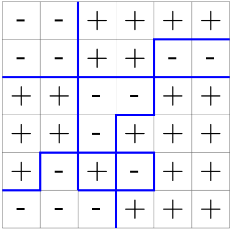


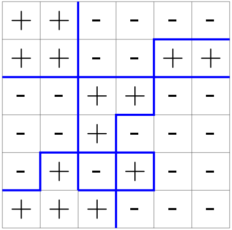
# Colorings and contours



# Colorings and contours

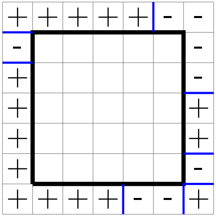




## Relation with the Ising model

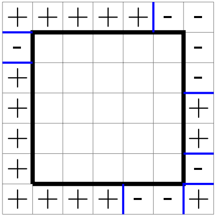
$$\beta = 1/T$$




$$\pi^{\beta}_{\Lambda,\eta}(\omega) = \frac{1}{Z} \exp\left(\beta \sum_{i \sim j,\ i \text{ or } j \in \Lambda} \omega_i \omega_j\right)$$

# Relation with the Ising model

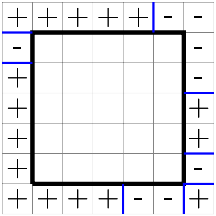
$$\beta = 1/T$$



$$\pi^{\beta}_{\Lambda,\eta}(\omega) = \frac{1}{Z}\exp\Big(\beta \sum_{i\sim j,\ i \text{ or } j\in\Lambda} \omega_i\omega_j\Big)$$

$$= \frac{1}{Z}\exp\Big(\beta\Big(N_g(\omega_\Lambda) - N_b(\omega_\Lambda)\Big)\Big)$$

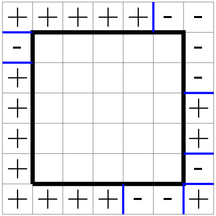
## Relation with the Ising model




$$\beta = 1/T$$

$$\begin{aligned}\pi^{\beta}_{\Lambda,\eta}(\omega) &= \frac{1}{Z}\exp\Big(\beta \sum_{i\sim j,\ i \text{ or } j\in\Lambda} \omega_i\omega_j\Big)\\ &= \frac{1}{Z}\exp\Big(\beta\Big(N_g(\omega_\Lambda) - N_b(\omega_\Lambda)\Big)\Big)\\ &= \frac{1}{Z'}\exp\Big(-2\beta N_b(\omega_\Lambda)\Big)\end{aligned}$$

## Relation with the Ising model




$$\beta = 1/T$$

$$\begin{aligned}\pi^{\beta}_{\Lambda,\eta}(\omega) &= \frac{1}{Z}\exp\Big(\beta \sum_{i\sim j,\ i \text{ or } j\in\Lambda} \omega_i\omega_j\Big) \\ &= \frac{1}{Z}\exp\Big(\beta\Big(N_g(\omega_\Lambda) - N_b(\omega_\Lambda)\Big)\Big) \\ &= \frac{1}{Z'}\exp\Big(-2\beta N_b(\omega_\Lambda)\Big)\end{aligned}$$

Even percolation $\leftrightsquigarrow$ Ising model

Parameter $p$ $\leftrightsquigarrow$ $\beta$ such that $\exp(-2\beta) = \dfrac{p}{1-p}$

## Relation with the Ising model

$\beta_c = \frac{1}{2}\log(1+\sqrt{2})$

$\beta \leq \beta_c$ : a unique Gibbs measure

$\beta > \beta_c$ : two extremal measures $\pi_\beta^+$ and $\pi_\beta^-$

## Relation with the Ising model

$\beta_c = \frac{1}{2}\log(1+\sqrt{2})$

$\beta \leq \beta_c$ : a unique Gibbs measure
$\beta > \beta_c$ : two extremal measures $\pi_\beta^+$ and $\pi_\beta^-$

### Proposition

There exists a unique even percolation measure $\mu_p$ on $\mathbb{Z}^2$: it is the image by the **contour** application of any Gibbs measure for the Ising model with parameter $\beta(p) = \frac{1}{2}\log\left(\frac{1-p}{p}\right)$.

| $p$ | $0$ | | $\frac{1}{2}$ | | $1$ |
|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | $\beta_c$ | $0$ | $-\beta_c$ | $-\infty$ |

## Relation with the Ising model

$\beta_c = \frac{1}{2}\log(1+\sqrt{2})$

$\beta \leq \beta_c$ : a unique Gibbs measure

$\beta > \beta_c$ : two extremal measures $\pi_\beta^+$ and $\pi_\beta^-$

**Proposition**

There exists a unique even percolation measure $\mu_p$ on $\mathbb{Z}^2$: it is the image by the **contour** application of any Gibbs measure for the Ising model with parameter $\beta(p) = \frac{1}{2}\log\left(\frac{1-p}{p}\right)$.

| $p$ | $0$ | | $\frac{1}{2}$ | | $1$ |
|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | $\beta_c$ | $0$ | $-\beta_c$ | $-\infty$ |

**Remark:** $\beta(1-p) = -\beta(p)$

Even perco $\mu_p$ $\leftrightsquigarrow$ Even perco $\mu_{1-p}$

Blue $\longleftrightarrow$ Grey

Ising parameter $\beta$ $\leftrightsquigarrow$ Ising parameter $-\beta$

Spin inversion on a checkerboard

## Number of infinite connected components

**Proposition**

Let $p \in [0,1]$, and $N =$ number of infinite connected components.

We have: $\mu_p(N = 0) = 1$ or $\mu_p(N = 1) = 1$.

## Number of infinite connected components

**Proposition**

Let $p \in [0,1]$, and $N =$ number of infinite connected components.

We have: $\mu_p(N = 0) = 1$ or $\mu_p(N = 1) = 1$.

1. By ergodicity, there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mu_p(N = k) = 1$.

## Number of infinite connected components

**Proposition**

Let $p \in [0,1]$, and $N =$ number of infinite connected components.
We have: $\mu_p(N=0) = 1$ or $\mu_p(N=1) = 1$.

1. By ergodicity, there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mu_p(N=k) = 1$.
2. We must have $k \in \{0, 1, \infty\}$, indeed:
   for $k \geq 2$, $\mu_p(N=k) > 0 \implies \mu_p(N=k-1) > 0$

## Number of infinite connected components

**Proposition**

Let $p \in [0,1]$, and $N =$ number of infinite connected components.
We have: $\mu_p(N=0) = 1$ or $\mu_p(N=1) = 1$.

1. By ergodicity, there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mu_p(N=k) = 1$.
2. We must have $k \in \{0, 1, \infty\}$, indeed:
   for $k \geq 2$, $\mu_p(N=k) > 0 \implies \mu_p(N=k-1) > 0$
3. Let us prove by contradiction that $\mu_p(N=\infty) = 0$.

## Number of infinite connected components

**Proposition**

Let $p \in [0,1]$, and $N =$ number of infinite connected components.
We have: $\mu_p(N=0)=1$ or $\mu_p(N=1)=1$.

1. By ergodicity, there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mu_p(N=k)=1$.
2. We must have $k \in \{0,1,\infty\}$, indeed:
   for $k \geq 2$, $\mu_p(N=k)>0 \implies \mu_p(N=k-1)>0$
3. Let us prove by contradiction that $\mu_p(N=\infty)=0$.
   1. Any point has a positive probability to be a **trifurcation**

## Number of infinite connected components

**Proposition**

Let $p \in [0,1]$, and $N =$ number of infinite connected components.
We have: $\mu_p(N=0) = 1$ or $\mu_p(N=1) = 1$.

1. By ergodicity, there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mu_p(N=k) = 1$.
2. We must have $k \in \{0, 1, \infty\}$, indeed:
   for $k \geq 2$, $\mu_p(N=k) > 0 \implies \mu_p(N=k-1) > 0$
3. Let us prove by contradiction that $\mu_p(N=\infty) = 0$.
   1. Any point has a positive probability to be a **trifurcation**
   2. In a box of size $L$, number of trifurcations $\propto L^2$.

## Number of infinite connected components

**Proposition**

Let $p \in [0,1]$, and $N =$ number of infinite connected components.
We have: $\mu_p(N=0) = 1$ or $\mu_p(N=1) = 1$.

1. By ergodicity, there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mu_p(N=k) = 1$.
2. We must have $k \in \{0, 1, \infty\}$, indeed:
   for $k \geq 2$, $\mu_p(N=k) > 0 \implies \mu_p(N=k-1) > 0$
3. Let us prove by contradiction that $\mu_p(N=\infty) = 0$.
   1. Any point has a positive probability to be a **trifurcation**
   2. In a box of size $L$, number of trifurcations $\propto L^2$.
   3. So, number of infinite connected components intersecting the box $\propto L^2$.

## Number of infinite connected components

**Proposition**

Let $p \in [0,1]$, and $N =$ number of infinite connected components. We have: $\mu_p(N=0) = 1$ or $\mu_p(N=1) = 1$.

1. By ergodicity, there exists $k \in \mathbb{N} \cup \{\infty\}$ such that $\mu_p(N=k) = 1$.
2. We must have $k \in \{0, 1, \infty\}$, indeed: for $k \geq 2$, $\mu_p(N=k) > 0 \implies \mu_p(N=k-1) > 0$
3. Let us prove by contradiction that $\mu_p(N=\infty) = 0$.
   1. Any point has a positive probability to be a **trifurcation**
   2. In a box of size $L$, number of trifurcations $\propto L^2$.
   3. So, number of infinite connected components intersecting the box $\propto L^2$.
   4. But that number cannot be larger than the perimeter of the box $\propto L$, contradiction!

## Number of infinite connected components

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}) \;\leftrightsquigarrow\; p_c = 1 - \frac{1}{\sqrt{2}} \;\left(< \frac{1}{2}\right)$$

## Number of infinite connected components

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}) \;\leftrightsquigarrow\; p_c = 1 - \frac{1}{\sqrt{2}} \;\left(< \frac{1}{2}\right)$$

**Proposition**

For the measure $\mu_p$ of even percolation:

- if $p < p_c$ $(\beta > \beta_c)$, a.s. no infinite connected component,
- if $p_c < p < 1/2$ $(0 < \beta < \beta_c)$, a.s. a (unique) infinite connected component.

| $p$ | $0$ | $p_c$ | $\frac{1}{2}$ | $1-p_c$ | $1$ |
|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | $\beta_c$ | $0$ | $-\beta_c$ | $-\infty$ |
| $\mu_p$ | no perco | perco | | | |

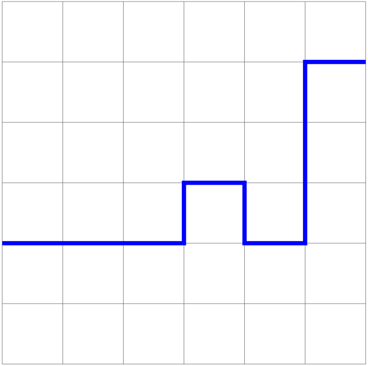
If $p < p_c$ ($\beta > \beta_c$), a.s. no infinite connected component.

If $p < p_c$ ($\beta > \beta_c$), a.s. no infinite connected component.

Assume there is an infinite path.

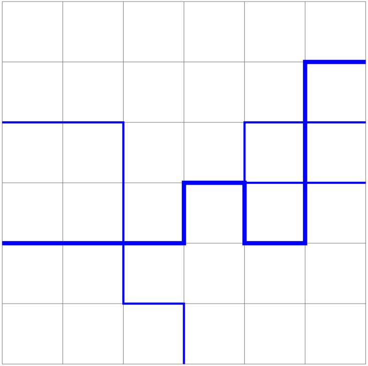
If $p < p_c$ ($\beta > \beta_c$), a.s. no infinite connected component.

Assume there is an infinite path.



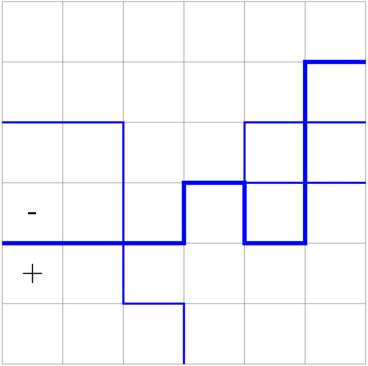
If $p < p_c$ ($\beta > \beta_c$), a.s. no infinite connected component.

Assume there is an infinite path.



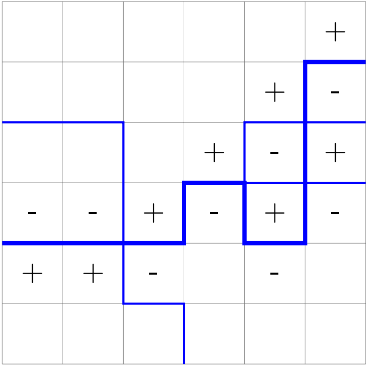
If $p < p_c$ $(\beta > \beta_c)$, a.s. no infinite connected component.

Assume there is an infinite path.



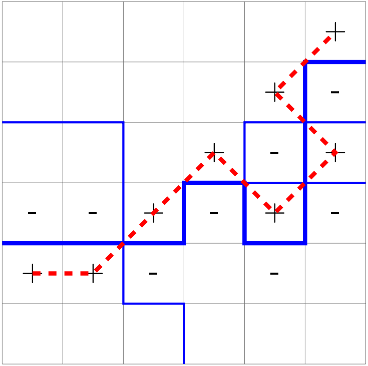
If $p < p_c$ ($\beta > \beta_c$), a.s. no infinite connected component.

Assume there is an infinite path.



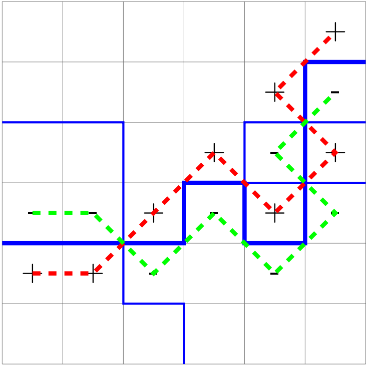
If $p < p_c$ ($\beta > \beta_c$), a.s. no infinite connected component.

Assume there is an infinite path.



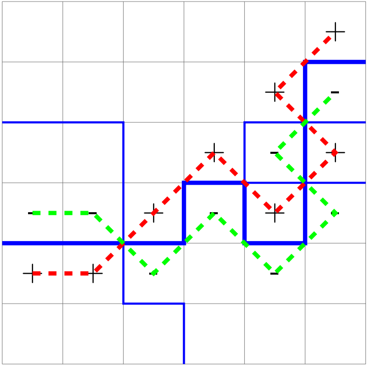
If $p < p_c$ ($\beta > \beta_c$), a.s. no infinite connected component.

Assume there is an infinite path.



If $p < p_c$ ($\beta > \beta_c$), a.s. no infinite connected component.

Assume there is an infinite path.




But we know that for $\beta > \beta_c$, under $\pi_\beta^+$, there are no infinite $*$-paths of spins $-$, contradiction!

[Russo 1979]

If $p_c < p < 1/2$ ($0 < \beta < \beta_c$), a.s. a (unique) infinite connected component.

If $p_c < p < 1/2$ $(0 < \beta < \beta_c)$, a.s. a (unique) infinite connected component.

For $0 < \beta < \beta_c$ :

- there is an infinite $*$-path of spins $+$,
- all the connected components of spins $+$ are finite.

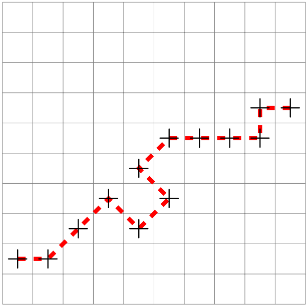
[Coniglia et al. 1976, Higuchi 1993]

If $p_c < p < 1/2$ $(0 < \beta < \beta_c)$, a.s. a (unique) infinite connected component.

For $0 < \beta < \beta_c$ :

- there is an infinite $*$-path of spins $+$,
- all the connected components of spins $+$ are finite.

[Coniglia et al. 1976, Higuchi 1993]

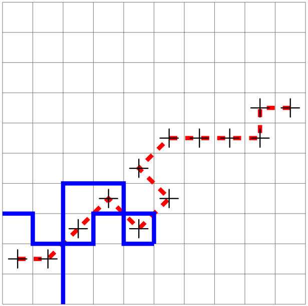


If $p_c < p < 1/2$ $(0 < \beta < \beta_c)$, a.s. a (unique) infinite connected component.

For $0 < \beta < \beta_c$ :

- there is an infinite $*$-path of spins $+$,
- all the connected components of spins $+$ are finite.

[Coniglia et al. 1976, Higuchi 1993]

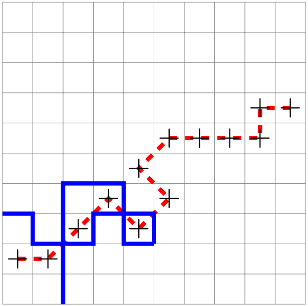


If $p_c < p < 1/2$ ($0 < \beta < \beta_c$), a.s. a (unique) infinite connected component.

For $0 < \beta < \beta_c$ :

- there is an infinite $*$-path of spins $+$,
- all the connected components of spins $+$ are finite.

[Coniglia et al. 1976, Higuchi 1993]



The union of contours of connected components of spins $+$ provides an infinite connected component for the even perco.

## Summary

| | | | | |
|---|---|---|---|---|
| $p$ | 0 | $p_c$ | $\frac{1}{2}$ | 1 |
| $\beta(p)$ | $+\infty$ | $\beta_c$ | 0 | $-\infty$ |
| $\mu_p$ | no perco \| | perco \| | | |

# Summary

| | | | | | |
|---|---|---|---|---|---|
| $p$ | $0$ | $p_c$ | $\frac{1}{2}$ | $1-p_c$ | $1$ |
| $\beta(p)$ | $+\infty$ | $\beta_c$ | $0$ | $-\beta_c$ | $-\infty$ |
| $\mu_p$ | no perco | perco | | | perco |

# Random Cluster model

- On a finite graph $G = (V, E)$, distribution:

$$\phi_{p,q}(\omega) = \frac{1}{Z} p^{N_b(\omega)} (1-p)^{N_g(\omega)} q^{k(\omega)},$$

where $k(\omega)$ = number of blue connected components.

# Random Cluster model

- On a finite graph $G = (V, E)$, distribution:

$$\phi_{p,q}(\omega) = \frac{1}{Z} p^{N_b(\omega)} (1-p)^{N_g(\omega)} q^{k(\omega)},$$

where $k(\omega) =$ number of blue connected components.

- Extension to an infinite volume measure on $\mathbb{Z}^2$, for $q \geq 1$. Critical point for the emergence of an infinite connected component: $p_c^{RC} = \frac{\sqrt{q}}{1+\sqrt{q}}$ [Beffara, Duminil-Copin 2012]

## Random Cluster model

- On a finite graph $G = (V, E)$, distribution:

$$\phi_{p,q}(\omega) = \frac{1}{Z} p^{N_b(\omega)} (1-p)^{N_g(\omega)} q^{k(\omega)},$$

where $k(\omega)$ = number of blue connected components.

- Extension to an infinite volume measure on $\mathbb{Z}^2$, for $q \geq 1$.
Critical point for the emergence of an infinite connected component: $p_c^{RC} = \frac{\sqrt{q}}{1+\sqrt{q}}$ [Beffara, Duminil-Copin 2012]

| | | |
|---|---|---|
| Ising model | $\leftrightsquigarrow$ | Random Cluster model |
| Parameter $\beta$ | $\leftrightsquigarrow$ | Parameters $p = f(\beta) = 1 - \exp(-2\beta)$, $q = 2$ |

To obtain the Random Cluster model from the Ising model, keep each edge between identical spins with probability $f(\beta) = 1 - \exp(-2\beta)$, independently.

# Ising model

| | | | | | |
|---|---|---|---|---|---|
| + | + | - | - | - | - |
| + | + | - | - | + | + |
| - | - | + | + | - | - |
| - | - | + | - | - | - |
| - | + | - | + | - | - |
| + | + | + | - | - | - |

$$\gamma_\beta(p)$$

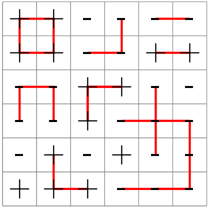
Ising model

| | | | | | |
|---|---|---|---|---|---|
| + | + | - | - | - | - |
| + | + | - | - | + | + |
| - | - | + | + | - | - |
| - | - | + | - | - | - |
| - | + | - | + | - | - |
| + | + | + | - | - | - |

$\gamma_{\beta(p)}$

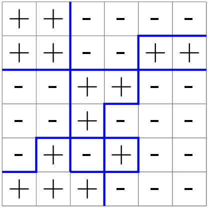
↙ ↘

$\varphi_{f(\beta(p)),2}$



Random cluster

$\mu_p$



Even percolation

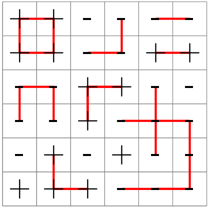
## Ising model

| | | | | | |
|---|---|---|---|---|---|
| + | + | - | - | - | - |
| + | + | - | - | + | + |
| - | - | + | + | - | - |
| - | - | + | - | - | - |
| - | + | - | + | - | - |
| + | + | + | - | - | - |

$\gamma_{\beta(p)}$

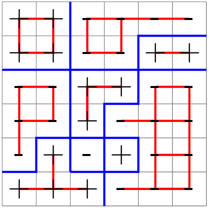
↙ ↘

$\varphi_{f(\beta(p)),2}$



Random cluster

$\mu_p$



Even percolation

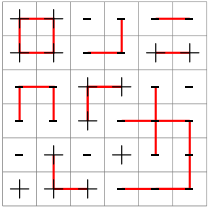
## Ising model

| | | | | | |
|---|---|---|---|---|---|
| + | + | - | - | - | - |
| + | + | - | - | + | + |
| - | - | + | + | - | - |
| - | - | + | - | - | - |
| - | + | - | + | - | - |
| + | + | + | - | - | - |

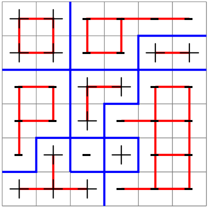
$\gamma_{\beta(p)}$

$\swarrow$ $\searrow$

$\varphi_{f(\beta(p)),2}$ $(\mu_p)_*$



Random cluster



Even percolation

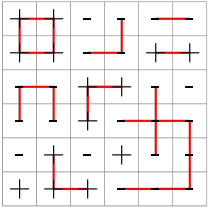
# Ising model

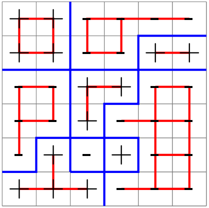
| | | | | | |
|---|---|---|---|---|---|
| + | + | - | - | - | - |
| + | + | - | - | + | + |
| - | - | + | + | - | - |
| - | - | + | - | - | - |
| - | + | - | + | - | - |
| + | + | + | - | - | - |

$\gamma_{\beta(p)}$

↙ ↘

$\varphi_{f(\beta(p)),2}$ $\preceq$ $(\mu_p)_*$





Random cluster — Even percolation

$$p \leq 1/2 \;\rightsquigarrow\; \beta(p) \geq 0$$

$$\varphi_{f(\beta(p)),2} \preceq (\mu_p)_*$$

$p \leq 1/2 \rightsquigarrow \beta(p) \geq 0$

$$\varphi_{f(\beta(p)),2} \preceq (\mu_p)_* \quad \Longrightarrow \quad \mu_p \preceq (\varphi_{f(\beta(p)),2})_*$$

$p \leq 1/2 \rightsquigarrow \beta(p) \geq 0$

$$\varphi_{f(\beta(p)),2} \preceq (\mu_p)_* \quad \Longrightarrow \quad \mu_p \preceq (\varphi_{f(\beta(p)),2})_* = \varphi_{2p,2}$$

$p \leq 1/2 \rightsquigarrow \beta(p) \geq 0$

$$\varphi_{f(\beta(p)),2} \preceq (\mu_p)_* \quad \Longrightarrow \quad \mu_p \preceq (\varphi_{f(\beta(p)),2})_* = \varphi_{2p,2}$$

$$\Longrightarrow \quad (\varphi_{2p,2})^c \preceq \mu_{1-p}$$

$p \leq 1/2 \rightsquigarrow \beta(p) \geq 0$

$$\varphi_{f(\beta(p)),2} \preceq (\mu_p)_* \quad \Longrightarrow \quad \mu_p \preceq (\varphi_{f(\beta(p)),2})_* = \varphi_{2p,2}$$

$$\Longrightarrow \quad (\varphi_{2p,2})^c \preceq \mu_{1-p}$$

Grey edges of $RC(2p, 2) \preceq$ Blue edges of $\mu_{1-p}$

$p \leq 1/2 \rightsquigarrow \beta(p) \geq 0$

$$\varphi_{f(\beta(p)),2} \preceq (\mu_p)_* \quad \Longrightarrow \quad \mu_p \preceq (\varphi_{f(\beta(p)),2})_* = \varphi_{2p,2}$$

$$\Longrightarrow \quad (\varphi_{2p,2})^c \preceq \mu_{1-p}$$

Grey edges of $RC(2p, 2) \preceq$ Blue edges of $\mu_{1-p}$

For $p < p_c^{RC} = \frac{\sqrt{2}}{1+\sqrt{2}}$, grey edges of $RC(p, 2)$ percolate.

$p \leq 1/2 \rightsquigarrow \beta(p) \geq 0$

$$\varphi_{f(\beta(p)),2} \preceq (\mu_p)_* \implies \mu_p \preceq (\varphi_{f(\beta(p)),2})_* = \varphi_{2p,2}$$

$$\implies (\varphi_{2p,2})^c \preceq \mu_{1-p}$$

Grey edges of $RC(2p,2) \preceq$ Blue edges of $\mu_{1-p}$

For $p < p_c^{RC} = \frac{\sqrt{2}}{1+\sqrt{2}}$, grey edges of $RC(p,2)$ percolate. (?)
[Beffara, Duminil-Copin 2012]

## Summary

| | | | | |
|---|---|---|---|---|
| $p$ | $0$ | $p_c$ | $\frac{1}{2}$ | $1$ |
| $\beta(p)$ | $+\infty$ | $\beta_c$ | $0$ | $-\infty$ |
| $\mu_p$ | no perco \| | perco \| | | |

## Summary

| $p$ | $0$ | $p_c$ | $\frac{1}{2}$ | $1-p_c$ | $1$ |
|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | $\beta_c$ | $0$ | $-\beta_c$ | $-\infty$ |
| $\mu_p$ | no perco | perco | | | perco |

## Summary

| | | | | | |
|---|---|---|---|---|---|
| $p$ | $0$ | $p_c$ | $\frac{1}{2}$ | $1 - p_c$ | $1$ |
| $\beta(p)$ | $+\infty$ | $\beta_c$ | $0$ | $-\beta_c$ | $-\infty$ |
| $\mu_p$ | no perco | perco | ? | perco | |

## Summary

| $p$ | $0$ | | $p_c$ | | $\frac{1}{2}$ | | $1 - p_c$ | | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | | $\beta_c$ | | $0$ | | $-\beta_c$ | | $-\infty$ |
| $\mu_p$ | | no perco | \| | perco | \| | ? | \| | perco | |

Unlike classical Bernoulli percolation,
monotony is not obvious for the even percolation!

# Summary

| $p$ | $0$ | | $p_c$ | | $\frac{1}{2}$ | | $1-p_c$ | | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | | $\beta_c$ | | $0$ | | $-\beta_c$ | | $-\infty$ |
| $\mu_p$ | | no perco | | perco | | ? | | perco | |

Unlike classical Bernoulli percolation,
monotony is not obvious for the even percolation!

But we are able to compare $\mu_p$ and $\mu_{1-p}$.

# Summary

| $p$ | $0$ | | $p_c$ | | $\frac{1}{2}$ | | $1-p_c$ | | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | | $\beta_c$ | | $0$ | | $-\beta_c$ | | $-\infty$ |
| $\mu_p$ | | no perco | \| | perco | \| | ? | \| | perco | |

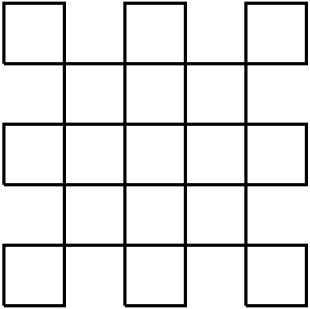
Unlike classical Bernoulli percolation,
monotony is not obvious for the even percolation!

But we are able to compare $\mu_p$ and $\mu_{1-p}$.

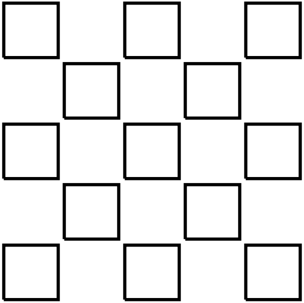
**Proposition**

If $p \leq 1/2$, the measure $\mu_p$ is "less connected" than $\mu_{1-p}$.

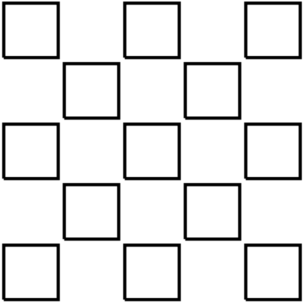
## Rejection sampling for the even percolation

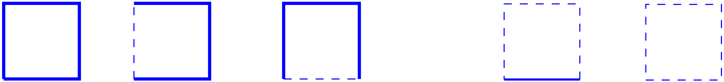


# Rejection sampling for the even percolation



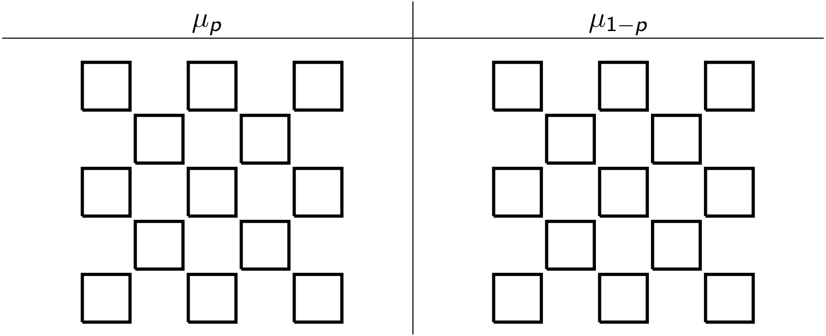
## Rejection sampling for the even percolation



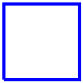


$p^4$ $\quad p^3(1-p)$ $\quad p^3(1-p)$ $\quad \cdots$ $\quad (1-p)^3 p$ $\quad (1-p)^4$

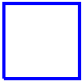
We build simultaneously a configuration distributed according to $\mu_p$ and one according to $\mu_{1-p}$.



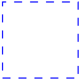
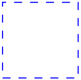
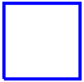
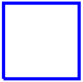
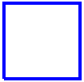
We build simultaneously a configuration distributed according to $\mu_p$ and one according to $\mu_{1-p}$.

| $\mu_p$ | | |
|---|---|---|
|  | | $(1-p)^4$ |
|  | | $p^4$ |

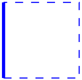
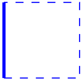
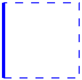
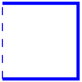
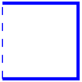
We build simultaneously a configuration distributed according to $\mu_p$ and one according to $\mu_{1-p}$.

| $\mu_p$ | $\mu_{1-p}$ | |
|---|---|---|
|  |  | $(1-p)^4$ |
|  |  | $p^4$ |

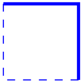
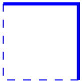
We build simultaneously a configuration distributed according to $\mu_p$ and one according to $\mu_{1-p}$.

| $\mu_p$ | $\mu_{1-p}$ | |
|---|---|---|
|  |  | $p^4$ |
|  |  | $(1-p)^4 - p^4$ |
|  |  | $p^4$ |

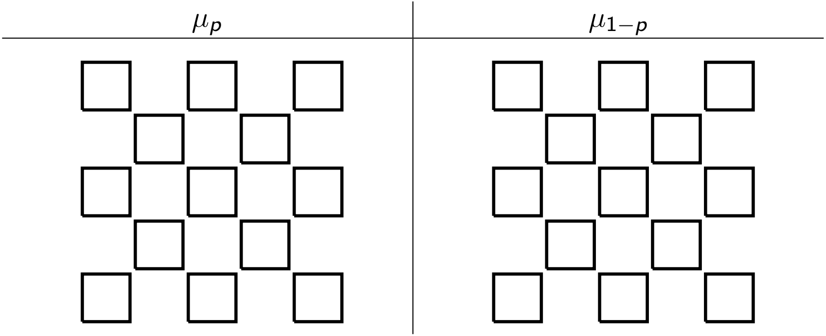
We build simultaneously a configuration distributed according to $\mu_p$ and one according to $\mu_{1-p}$.

| $\mu_p$ | $\mu_{1-p}$ | |
|---|---|---|
|  |  | $p^3(1-p)$ |
|  |  | $p(1-p)^3 - p^3(1-p)$ |
|  |  | $p^3(1-p)$ |

We build simultaneously a configuration distributed according to $\mu_p$ and one according to $\mu_{1-p}$.

| $\mu_p$ | $\mu_{1-p}$ | |
|---|---|---|
|  |  | $p^2(1-p)^2$ |

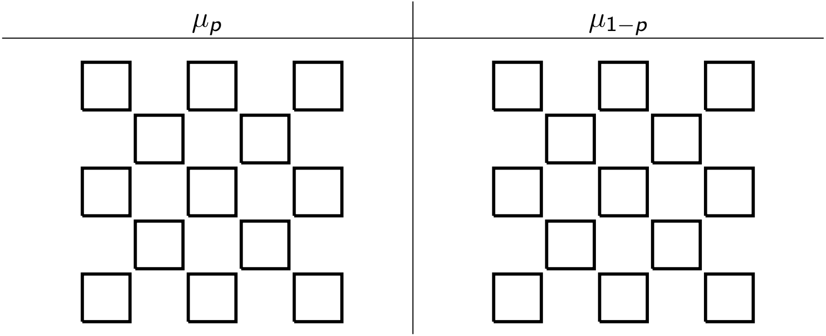
## Coupling between $\mu_p$ and $\mu_{1-p}$



For each elementary square:

- either all the edges are identical
- or all the edges are opposite.

# Coupling between $\mu_p$ and $\mu_{1-p}$

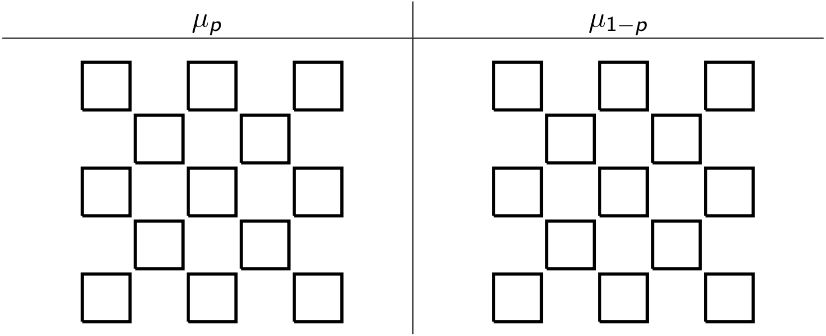



For each elementary square:

- either all the edges are identical
- or all the edges are opposite.

So, same parity at each point.

# Coupling between $\mu_p$ and $\mu_{1-p}$




For each elementary square:

- either all the edges are identical
- or all the edges are opposite.

So, same parity at each point.

And the configuration distributed according to $\mu_{1-p}$ is more connected than the one distributed according to $\mu_p$ !

## Summary

| $p$ | $0$ | | $p_c$ | | $\frac{1}{2}$ | | $1-p_c$ | | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | | $\beta_c$ | | $0$ | | $-\beta_c$ | | $-\infty$ |
| $\mu_p$ | | no perco | \| | | perco | | \| | perco | |

## Summary

| $p$ | $0$ | $p_c$ | $\frac{1}{2}$ | $1-p_c$ | $1$ |
|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | $\beta_c$ | $0$ | $-\beta_c$ | $-\infty$ |
| $\mu_p$ | no perco \| | | perco \| | | perco |

**Conjecture:** if $G$ is a finite Eulerian graph, the sequence of even percolation measures $(\mu_p)_{p\in[0,1]}$ is stochastically non-decreasing.

## Summary

| $p$ | $0$ | | $p_c$ | | $\frac{1}{2}$ | | $1-p_c$ | | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $\beta(p)$ | $+\infty$ | | $\beta_c$ | | $0$ | | $-\beta_c$ | | $-\infty$ |
| $\mu_p$ | | no perco | \| | | perco | | \| | perco | |

**Conjecture:** if $G$ is a finite Eulerian graph, the sequence of even percolation measures $(\mu_p)_{p\in[0,1]}$ is stochastically non-decreasing.

*Does Eulerian percolation on $\mathbb{Z}^2$ percolate?*
O. Garet, R. Marchand, I. Marcovici
`arXiv:1607.01974`