Universality of height fluctuations in the dimer model

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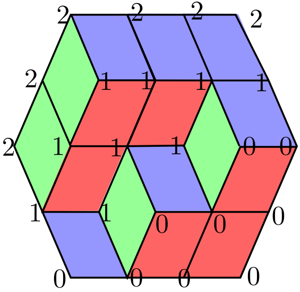
Weighted dimer model

A dimer configuration or perfect matching M of a graph G is a subset of edges such that every vertex is an endpoint of a single edge in M.
If G is a finite weighted unoriented graph, the dimer model denotes the measure

$$\mathbb{P}(M) \propto \prod_{e \in M} w(e).$$

We will always assume that G is a bipartite planar graph.
Notation : We write G^δ for a sequence of graphs with mesh size δ.

Height function

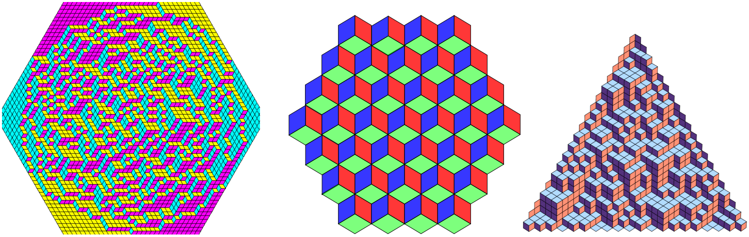


One can describe a tiling by a height function *h*, for example the *z* coordinate of the surface. Questions about the general shape of the surface translate directly into asymptotic of this height function.

When we scale the tiling by δ, we will keep the original scaling of the height function, i.e. taking values in $\mathbb{Z}$. We write $h^{\#\delta}$ for the height function when we want to emphasize the scale.

Planar boundary condition

Depending on the microscopic behaviour on the boundary, the height function and the tiling can change a lot.



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We will work with cases where the boundary lies in a plane, as in the middle and right hand side pictures.

Lozenge tiling result

We prove that there are planar domains of any slope and shape where we can describe the law of the fluctuations.

Theorem (Berestycki, L., Ray)

Let P be a plane in $\mathbb{R}^3$ and let λ be a closed simple loop in P. Then there exists a sequence of sub-graphs G^δ such that, as $\delta \to 0$, $\delta h^{\#\delta}(\partial G^\delta) \to \lambda$ as closed sets in $\mathbb{R}^3$ and

$$(h^{\#\delta} - \mathbb{E}(h^{\#\delta})) \circ \ell \to \frac{1}{2\pi\chi} h_{GFF},$$

where ℓ is an explicit linear map determined by P and h_{GFF} is a Gaussian free field with Dirichlet boundary conditions.

Uniform spanning tree

On a possibly weighted and oriented graph, a wired spanning tree is a subset of oriented edges such that each vertex has exactly one outgoing edge, with a sink on the boundary.

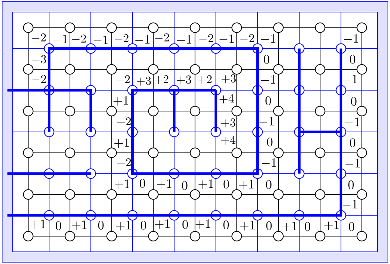
The uniform spanning tree measure is defined by $\mathbb{P}(\mathcal{T}) \propto \prod_{\vec{e} \in \mathcal{T}} w(\vec{e})$.

One can sample from that measure by Wilson's algorithm :

- Pick any vertex v.
- Sample a random walk from v.
- Erase all its loops chronologically. This gives the path γ_v from v to the boundary.
- Add γ_v to the boundary and iterate.

Temperley's bijection

There exists a bijection between the dimer model on a graph G_{dimers} and the uniform spanning tree on a related graph G_{tree}.



The height function in the dimer model is the winding of the UST branches :

$$h(v) = c\text{Winding}(\gamma_v).$$

Main result

Take G_{tree} to be an infinite planar graph satisfying the assumptions below.

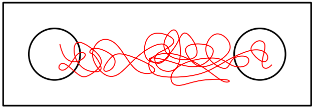
Theorem (Berestycki, L., Ray)

Let D be a simply connected domain with C^1 boundary. Let $\mathcal{T}^{\#\delta}$ be a uniform (wired) spanning tree on $(\delta G) \cap D$, and for any $v \in D^{\#\delta}$ let $h^{\#\delta}(v)$ denote the total winding of the branch of $\mathcal{T}^{\#\delta}$ connecting v and ∂D. We have

$$h^{\#\delta} - \mathbb{E}(h^{\#\delta}) \to \frac{1}{\chi} h^0_{GFF}.$$

Assumptions on the graph

- **Planarity** : The graph G_{tree} is embedded in the plane with non-crossing edges.
- **Central limit theorem** : $\frac{X^0_{nt}}{\sqrt{n}} \to B_t$ up to a time change.
- **Uniform crossing** : The probability to cross a rectangle is uniformly bounded away from 0.

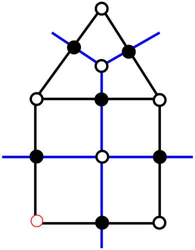


- **Technicalities** : The density of vertices is unifomly bounded. The winding of an edge is uniformly bounded. The random walk is irreducible.

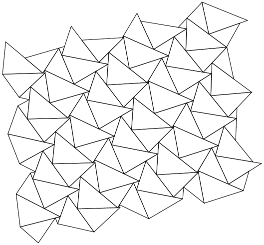
No assumption of periodicity or even ergodicity.

Universality

To go back to the dimer model, we need to look at the relation between the graphs G_{dimers} and G_{tree}.



From any G_{tree}, one can obtain G_{dimers} as $G_{\text{tree}} \cup G^*_{\text{tree}}$.



There is a construction from any G_{dimers} but the resulting G_{tree} is a bit nasty.

Initial setup

What is known :

- Scaling limit of the UST
- A continuous version of the bijection
- The winding is almost a continuous function of a path.

Difficulties :

- The winding is almost a continuous function of a path.
- The continuous bijection is not phrased in term of actual winding.
- Everything diverges pointwise.
- What is even the winding of an open path ?

Winding of an open path

Let $\gamma : [0,1] \mapsto \mathbb{C}$ be a smooth path. Two definitions of winding !

- $W(\gamma) = \arg(\gamma'(1)) - \arg(\gamma'(0))$
- $W(\gamma, z) = \arg(\gamma(1) - z) - \arg(\gamma(0) - z)$

Hopefully they are related

Proposition

If γ is self avoiding, then

$$W(\gamma) = W(\gamma, \gamma(1)) - W(\gamma, \gamma(0)).$$

Gaussian free field

The gaussian free field in a domain D (with Dirichlet boundary condition) is the random centred Gaussian function defined by the covariance :

$$\mathrm{Cov}(x,y) = \mathbb{E}[h(x)h(y)] = G_D(x,y),$$

where G_D is the Green function in D.

Not a function

It is not defined pointwise. It actually diverges logarithmically :

$$\int_{B(x,\varepsilon)} h \sim N(0, c\log\frac{1}{\varepsilon}).$$

n-point function

Still almost a function :

For all $x_1, \dots, x_n$, distinct, the n-point function $\mathbb{E} \prod h(x_1)$ is finite and well defined.

Essentially everything you want to write using it is true, for example :

$$\mathbb{E}\langle h, f\rangle^2 = \int\int \mathbb{E}[h(x)h(y)]f(x)f(y)dxdy.$$

- We prove convergence of these functions for all n and this identifies the moments.
- Today we will only look at the 2 point case.

$\rightarrow$ We only care about two curves !

Regularisation

We decompose the winding as follows. For every point v, consider the path γ_v from v to the boundary and parametrise it by capacity. We write

$$h_T(v) = \text{Winding}(\gamma_v(0, T), v) \qquad h(v) = h_T(v) + \varepsilon(v).$$

Advantages :

- h_T is a continuous function of the UST.
- ε has the name of something small. It only depends on a neighbourhood of v.

To do

- Rephrase the continuum results in term of h_T.
- Prove that the ε terms are irrelevant.

Local coupling

The ε terms will be irrelevant in the limit because they become independent.

Theorem (Berestycki, L., Ray)

Let D be a domain and $v_1, \dots, v_n \in D$. Let r be the minimal distance between the v_i and ∂D. There exists a coupling of

- $\mathcal{T}$ *a wired UST in* $(\delta G) \cap D$
- $\tilde{\mathcal{T}}_1, \dots, \tilde{\mathcal{T}}_n$ *independent whole plane USTs.*
- $c_1, \dots, c_n$ *stricly positive random variables*

such that

$$\mathcal{T} \cap B(v_i, c_i r) = \tilde{\mathcal{T}}_i \cap B(v_i, c_i r)$$

The law of the c_i only depends on the constants in the crossing estimate and they have a polynomial tail in 0.

An a priori distance estimate

Proposition

Let D be a domain and let $u, v \in D$. Let $r = |u - v| \wedge d(v, \partial D) \wedge d(u, \partial D)$. Let γ be a loop erased walk starting from v until it exits $(\delta G) \cap D$. Then for all $c \in (0, 1)$ such that $cr \geq \delta$,

$$\mathbb{P}(|\gamma - u| < cr) \leq Kc^{\alpha}$$

for some universal constants $K, \alpha > 0$.

This is completely uniform on the domain so we can think of D as being infinite and it still works.

Schramm-Loewner Evolution

I will not need much about SLE here so I only recall a few properties of radial SLE_2 :

- It is a simple curve between a boundary point and an interior point.
- It is the scaling limit of Loop-erased random walk, i.e tree branches.
- It is conformally invariant : The image of an SLE by a conformal map is still SLE.
- One can actually compute pretty well with it.

Continuum spanning tree

It exists. We only care about the joint law of 2 branches.

To sample the two branches from x and y :

- First sample SLE_2 from x to a point chosen according to the harmonic measure in D. Call it γ_x.
- Then sample SLE_2 in $D \setminus \gamma_x$ from y to a point chosen according to the harmonic measure in D.

In the second step, one can see it as sampling an independent SLE in the unit disc and mapping it to the domain $D \setminus \gamma_x$.

Imaginary geometry

It is the continuum analogue of Temperley's bijection in the $\kappa = 2$ case. We will not talk about other values of κ.

It is phrased as the existence of a coupling between SLE curves and a GFF such that when we condition by one curve γ we obtain :

$$h|\gamma \overset{\text{law}}{=} (\text{GFF in } D \setminus \gamma) + (\text{harmonic function})$$

The harmonic function is $\arg g'$ where g is a conformal map sending $D \setminus \gamma$ to the disc.

Conformal covariance of winding

The connection between the above definition and the winding actually follows from a deterministic change of variable formula.

Lemma

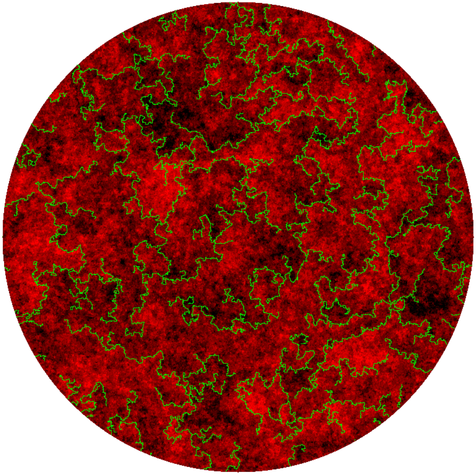
Let γ be a curve from 0 to 1 in the disc and g a conformal map preserving a neighbourhood of 1.

$$Winding(g(\gamma)) - Winding(\gamma) = \arg g'(0).$$

This is the key to go from our definition of a regularised winding function h_T to the imaginary geometry setting.

Open questions

- What happens with interacting dimers ?
- What happens in more general topology ?
  Work in progress with N. Berestycki and G. Ray.
- Can the technique be used in a non-planar setting ?
- What happen to the tree when there is a gaseous phase ?



Thank you for your attention.