Plancherel measures on strict partitions: Polynomiality and limit shape problems

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Overview

- The Ordinary Case Plancherel measure $\mathbb{P}_n^{\mathrm{Plan}}$ on (all) partitions of n:

$$\mathbb{P}_n^{\mathrm{Plan}}(\{\lambda\}) = \frac{(f^\lambda)^2}{n!}.$$

- The Strict Case Shifted Plancherel measure $\mathbb{P}_n^{\mathrm{SPl}}$ on strict partitions of n:

$$\mathbb{P}_n^{\mathrm{SPl}}(\{\lambda\}) = \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!}.$$

  (A partition λ is said to be strict if all parts of λ are pairwise distinct.)

Contents of this talk.

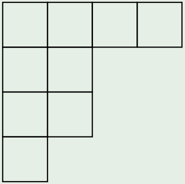
- Main Topic: Polynomiality theorems of shifted Plancherel averages (previous works due to Stanley (2010), Olshanski (2010), Okada, Panova (2012), M-Novak (2013), Han-Xiong (2016), . . .)
- Discussion: Limit Shape Problems, working in progress with Tomoyuki Shirai (Kyushu Univ.)

Plancherel measures

$\mathcal{P}_n$: the set of (all) partitions of n.
$Y(\lambda)$: the Young diagram of $\lambda \in \mathcal{P}_n$.

Example (Young diagram and standard Young tableau)

$\lambda = (4, 2, 2, 1)$, $Y(\lambda) =$



a SYT :

1	2	6	7
3	5		
4	9		
8			

Definition (Plancherel measure on $\mathcal{P}_n$)

$$\mathbb{P}_n^{\mathrm{Plan}}(\{\lambda\}) := \frac{(f^\lambda)^2}{n!} \qquad (\lambda \in \mathcal{P}_n).$$

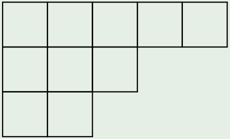
where f^λ is the number of standard Young tableaux (SYT) of shape $Y(\lambda)$.

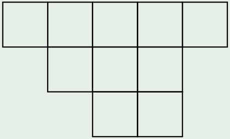
Strict partitions

- A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is called strict if all parts $\lambda_i (> 0)$ are distinct.
- $\mathcal{SP}_n$: the set of all strict partitions of n.

Example (shifted Young diagram and standard Young tableau)

$\lambda = (5, 3, 2) \in \mathcal{SP}_{10}$.

$Y(\lambda) =$ 

$S(\lambda) =$ 

a SYT of shape $S(\lambda)$:

1	2	4	5	8
	3	6	9	
		7	10	

Plancherel measures on strict partitions

$\mathcal{SP}_n$: the set of all strict partitions of n.

Definition (Plancherel measure on $\mathcal{SP}_n$)

$$\mathbb{P}_n^{\mathrm{SPl}}(\{\lambda\}) := \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \qquad (\lambda \in \mathcal{SP}_n).$$

where g^λ is the number of SYT of shape $S(\lambda)$.
It is called the Plancherel measure on strict partitions or shifted Plancherel measure.

This is indeed a *probability measure* on $\mathcal{SP}_n$ by virtue of the identity

$$\sum_{\lambda \in \mathcal{SP}_n} 2^{n-\ell(\lambda)}(g^\lambda)^2 = n!.$$

This identity can be proved by using projective representation theory of symmetric groups or shifted RSK correspondences (Sagan (1987), Worley (1984)).

$\mathbb{P}_n^{\mathrm{Plan}}$ versus $\mathbb{P}_n^{\mathrm{SPl}}$

In the introduction, we will compare two probablities $\mathbb{P}_n^{\mathrm{Plan}}$ and $\mathbb{P}_n^{\mathrm{SPl}}$.

- The Ordinary Case Plancherel measure on $\mathcal{P}_n$, all partitions of n:

$$\mathbb{P}_n^{\mathrm{Plan}}(\{\lambda\}) = \frac{(f^\lambda)^2}{n!}.$$

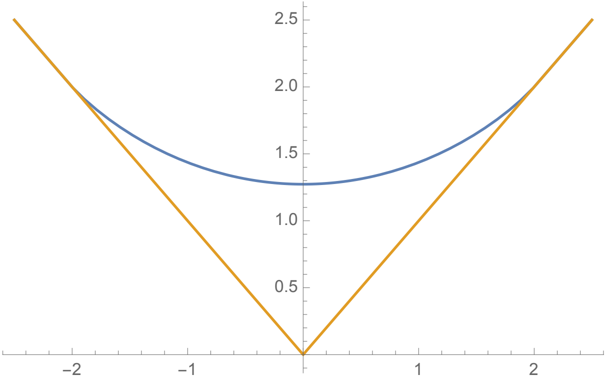
- The Strict Case Plancherel measure on $\mathcal{SP}_n$, strict partitions of n:

$$\mathbb{P}_n^{\mathrm{SPl}}(\{\lambda\}) = \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!}.$$

- Comparisons
  - Limit shape problem (Logan-Shepp-Vershik-Kerov Theorem)
  - Fluctuations of row-lengths λ_i (Baik-Deift-Johansson Theorem)

Logan-Shepp-Vershik-Kerov Limit Shape

The Ordinary Case Limit shape (Russian style)

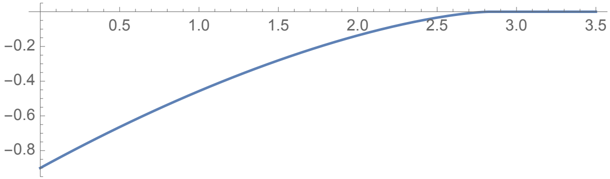


$$y = \Omega(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin \frac{x}{2} + \sqrt{4 - x^2}\right) & (|x| \leq 2) \\ |x| & (|x| \geq 2). \end{cases}$$

Note that $\Omega'(x) = \frac{2}{\pi} \arcsin \frac{x}{2}$ $(|x| \leq 2)$.

The Strict Case

Conjecture [Ivanov (2006), Bernstein-Henke-Regev (2007)].
English style, unshifted Young diagram $Y(\lambda)$



$$y = \Omega_{\mathrm{strict}}(x) = \begin{cases} \frac{1}{\pi}\left(x \arccos \frac{x}{2\sqrt{2}} - \sqrt{8-x^2}\right) & (0 \le x \le 2\sqrt{2}) \\ 0 & (x > 2\sqrt{2}). \end{cases}$$

Note that $\Omega'_{\mathrm{strict}}(x) = \frac{1}{\pi} \arccos \frac{x}{2\sqrt{2}}$ $(0 \le x \le 2\sqrt{2})$ and $\Omega_{\mathrm{strict}}(0) = -\frac{2\sqrt{2}}{\pi} = -0.90\ldots$.

Limit distribution for λ_i (The Ordinary Case)

$$\mathbb{E}_n^{\mathrm{Plan}}[\lambda_1] \quad \sim \quad 2\sqrt{n} \qquad (n \to \infty).$$

Theorem (Baik-Deift-Johansson (1999))

$$\lim_{n\to\infty} \mathbb{P}_n^{\mathrm{Plan}} \left(\frac{\lambda_1 - 2\sqrt{n}}{n^{1/6}} \le x \right) = F_{\mathrm{GUE}}(x),$$

where F_{GUE} is the Gaussian Unitary Ensemble Tracy-Widom distribution.

Theorem (Borodin-Okounkov-Olshanski, Johansson, Okounkov (2000))

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be distributed w.r.t. $\mathbb{P}_n^{\mathrm{Plan}}$. The random variables

$$\frac{\lambda_i - 2\sqrt{n}}{n^{1/6}}, \qquad i = 1, 2, \dots$$

converge to the Airy ensemble, in joint distribution.

Limit distribution for λ_i (The Strict Case)

$$\mathbb{E}_n^{\mathrm{SPl}}[\lambda_1] \quad \sim \quad 2\sqrt{2n} \qquad (n \to \infty).$$

Theorem (Tracy-Widom (2004))

$$\lim_{n\to\infty} \mathbb{P}_n^{\mathrm{SPl}} \left(\frac{\lambda_1 - 2\sqrt{2n}}{(2n)^{1/6}} \le x \right) = F_{\mathrm{GUE}}(x).$$

Theorem (M (2005))

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be distributed w.r.t. $\mathbb{P}_n^{\mathrm{SPl}}$. The random variables

$$\frac{\lambda_i - 2\sqrt{2n}}{(2n)^{1/6}}, \qquad i = 1, 2, \dots$$

converge to the Airy ensemble, in joint distribution.

Stanley's polynomiality theorem (The Ordinary Case)

$\Lambda = \Lambda_{\mathbb{Q}}$: the algebra of symmetric functions with $\mathbb{Q}$-coefficients.

Theorem (Stanley, Olshanski (2010))

For any symmetric function $f \in \Lambda$,

$$\sum_{\lambda \in \mathcal{P}_n} \frac{(f^\lambda)^2}{n!} f(\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_n - n)$$

is a polynomial in n.

For example, if $f = p_1 = x_1 + x_2 + \cdots$,

$$\sum_{\lambda \in \mathcal{P}_n} \frac{(f^\lambda)^2}{n!} p_1(\lambda_1 - 1, \lambda_2 - 2, \ldots, \lambda_n - n) = \sum_{\lambda \in \mathcal{P}_n} \frac{(f^\lambda)^2}{n!} \sum_{i=1}^{n} (\lambda_i - i)$$

$$= -\frac{n(n-1)}{2}.$$

Main polynomiality theorem (The Strict Case)

Recall $\Lambda = \mathbb{Q}[p_1, p_2, p_3, p_4, \ldots]$, where

$$p_k(x_1, x_2, \ldots) = \sum_{i \geq 1} x_i^k \qquad \text{(power-sum)}$$

Define the subalgebra $\Gamma = \mathbb{Q}[p_1, p_3, p_5, \ldots]$.

Main Polynomiality Theorem ([M])

For any $f \in \Gamma$, the shifted Plancherel average

$$\mathbb{E}_n^{\mathrm{SPl}}[f] := \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} f(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$$

is a polynomial in n.

Remark: We can not replace Γ by Λ.

Main polynomiality theorem (The Strict Case)

Example

$$\mathbb{E}_n^{\mathrm{SPl}}[p_1] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \sum_{i=1}^{\ell(\lambda)} \lambda_i = n.$$

$$\mathbb{E}_n^{\mathrm{SPl}}[p_3] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \sum_{i=1}^{\ell(\lambda)} \lambda_i^3 = 6\binom{n}{2} + \binom{n}{1}.$$

$$\mathbb{E}_n^{\mathrm{SPl}}[p_5] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \sum_{i=1}^{\ell(\lambda)} \lambda_i^5 = 80\binom{n}{3} + 30\binom{n}{2} + \binom{n}{1}.$$

(The first identity is trivial because $p_1(\lambda) = p_1(\lambda_1, \dots, \lambda_l) = \sum_i \lambda_i = |\lambda| = n$ for $\lambda \in \mathcal{SP}_n$.)

Main polynomiality theorem (The Strict Case)

Remark

The Plancherel average

$$\mathbb{E}_n^{\mathrm{SPl}}[p_2] = \sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} \sum_{i=1}^{\ell(\lambda)} \lambda_i^2$$

is NOT a polynomial in n.

$$\mathbb{E}_n^{\mathrm{SPl}}[p_1] = n, \qquad \mathbb{E}_n^{\mathrm{SPl}}[p_3] = 3n^2 - 2n.$$

Conjecture ([M])

$$\mathbb{E}_n^{\mathrm{SPl}}[p_2] \sim \frac{32\sqrt{2}}{9\pi} n^{\frac{3}{2}} \fallingdotseq 1.60 n^{\frac{3}{2}}.$$

(We will observe this conjecture in the end of this talk.)

Schur Q-functions

We show "Main Polynomiality Theorem". Our proof is exactly similar to [Olshanski (2010)].

Definition (Schur P-functions and Q-functions)

Let λ be a strict partition of length $l := \ell(\lambda) \leq N$. The Schur P-polynomial in N variables is defined by

$$P_{\lambda|N}(x_1, \dots, x_N) = \frac{1}{(N-l)!} \sum_{w \in \mathfrak{S}_N} w \left(x_1^{\lambda_1} \cdots x_l^{\lambda_l} \prod_{i=1}^{l} \prod_{j=i+1}^{N} \frac{x_i + x_j}{x_i - x_j} \right).$$

The Schur P-function P_λ is defined by the projective limit $P_\lambda = \varprojlim P_{\lambda|N}$.

The Schur Q-function Q_λ is defined by $Q_\lambda = 2^{\ell(\lambda)} P_\lambda$.

It is well known that

$$\Gamma = \mathbb{Q}[p_1, p_3, p_5, \dots] = \bigoplus_{n=0}^{\infty} \bigoplus_{\lambda \in \mathcal{SP}_n} \mathbb{Q} P_\lambda = \bigoplus_{n=0}^{\infty} \bigoplus_{\lambda \in \mathcal{SP}_n} \mathbb{Q} Q_\lambda.$$

Factorial Schur Q-functions

Definition (Okounkov, Ivanov (1999))

Let λ be a strict partition of length $l := \ell(\lambda) \leq N$. The factorial Schur P-polynomial in N variables is defined by

$$P^*_{\lambda|N}(x_1, \dots, x_N) = \frac{1}{(N-l)!} \sum_{w \in \mathfrak{S}_N} w \left(x_1^{\downarrow \lambda_1} \cdots x_l^{\downarrow \lambda_l} \prod_{i=1}^{l} \prod_{j=i+1}^{N} \frac{x_i + x_j}{x_i - x_j} \right),$$

where we set

$$x^{\downarrow k} = x(x-1)(x-2)\cdots(x-k+1).$$

The factorial Schur P-function P^*_λ is defined by their projective limit $P^*_\lambda = \varprojlim P^*_{\lambda|N}$.

The factorial Schur Q-function Q^*_λ is defined by $Q^*_\lambda = 2^{\ell(\lambda)} P^*_\lambda$.

$P^*_\lambda \in \Gamma$ is not homogeneous. The highest degree term of P^*_λ is P_λ.

Proof

Main Polynomiality Theorem ([M])

$$\mathbb{E}_n^{\mathrm{SPl}}[f] := \sum_{\lambda \in \mathcal{SP}_n} \mathbb{P}_n^{\mathrm{SPl}}[\lambda] f(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

is a polynomial in n for any $f \in \Gamma$.

Proof:

- $\{P_\nu^*\}_{\nu \in \mathcal{SP}}$ form a linear basis of Γ.
- Therefore it is sufficient to prove that $\mathbb{E}_n^{\mathrm{SPl}}[P_\nu^*]$ is a polynomial in n for each strict partition ν.
- In this case, we can obtain the explicit expression

$$\mathbb{E}_n^{\mathrm{SPl}}[P_\nu^*] = 2^{k-\ell(\nu)} g^\nu \binom{n}{k} \qquad (\nu \in \mathcal{SP}_k),$$

which is proved by using (well-studied) orthogonality relations for Schur Q-functions.

First degree estimate

For a given function $f \in \Gamma$, we want to estimate the degree of the polynomial $\mathbb{E}_n^{\mathrm{SPl}}[f]$ in n.

First, we consider the standard degree filtration of symmetric functions (as a "polynomial" in variables $x_1, x_2, \dots$).

$$\deg p_k = k \qquad (k = 1, 2, 3, \dots).$$

$(\deg x_i = 1)$

Proposition (A simple estimate)

If $\deg f = k$, then the degree of the polynomial $\mathbb{E}_n^{\mathrm{SPl}}[f]$ is at most k.

Proof:

- For the factorial Schur P-function P_ν^*, we have $\deg P_\nu^* = |\nu| = \sum_i \nu_i$.
- We showed $\mathbb{E}_n^{\mathrm{SPl}}[P_\nu^*] = 2^{|\nu| - \ell(\nu)} g^\nu \binom{n}{|\nu|}$. In particular, the polynomial $\mathbb{E}_n^{\mathrm{SPl}}[P_\nu^*]$ has degree $|\nu|$.

Second degree estimate

Proposition (A simple estimate)

If $\deg f = k$, then the degree of the polynomial $\mathbb{E}_n^{\mathrm{SPl}}[f]$ is at most k.

However, this degree estimate is not sharp in some cases:

$$\deg p_3 = 3 \qquad \text{but} \qquad \mathbb{E}_n^{\mathrm{SPl}}[p_3] = 6\binom{n}{2} + \binom{n}{1} = 3n^2 - 2n.$$

We now introduce the second degree filtration $\deg'$ on Γ by

$$\deg' p_{2k-1} = k.$$

Theorem (Han-Xiong (2016), [M])

If $\deg' f = k$, then the degree of the polynomial $\mathbb{E}_n^{\mathrm{SPl}}[f]$ is at most k.

A new identity

Theorem ([M] conjectured by Soichi Okada)

Let $r \geq 1$ and define the function φ_r on $\mathcal{SP}$ by

$$\varphi_r(\lambda) = \sum_{i=1}^{l(\lambda)} \prod_{k=-r}^{r} (\lambda_i + k).$$

Then we have

$$\mathbb{E}_n^{\mathrm{SPl}}[\varphi_r] = \frac{2^r (2r+1)!}{(r+1)!} \binom{n}{r+1}.$$

Proof:

- It is easy to see that $\varphi_r = \sum_{j=0}^{r} (-1)^{r-j} e_{r-j}(1^2, 2^2, \ldots, r^2) p_{2j+1}$.
- In particular, $\varphi_r \in \Gamma$ and $\deg'(\varphi_r) = r + 1$.
- Therefore $\mathbb{E}_n^{\mathrm{SPl}}[\varphi_r]$ is a polynomial in n of degree (at most) $r + 1$.
- Check that both sides coincide if $n = 0, 1, \ldots, r, r + 1$.

Stanley's polynomiality theorem 2 (The Ordinary Case)

Definition

For each box u in a Young diagram $Y(\lambda)$, we define its content $c_u \in \mathbb{Z}$ by

0	1	2	3	4
−1	0	1	2	
−2	−1	0	1	
−3	−2			

Theorem (Stanley, Olshanski (2010))

For any symmetric function $f \in \Lambda$,

$$\sum_{\lambda \in \mathcal{P}_n} \frac{(f^\lambda)^2}{n!} f(c_u \ : \ u \in Y(\lambda))$$

is a polynomial in n.

A formula for content evaluations

Theorem [Fujii-Kanno-Moriyama-Okada (2008)]

For each $r \geq 1$, define a symmetric function U_r by

$$U_r(x_1, x_2, \dots) = \sum_{j \geq 1} \prod_{i=0}^{r-1} (x_j^2 - i^2).$$

Then

$$\sum_{\lambda \in \mathcal{P}_n} \frac{(f^\lambda)^2}{n!} U_r(c_u \ : \ u \in Y(\lambda)) = \frac{(2r)!}{(r+1)!} \binom{n}{r+1}$$

$$U_1 = \sum_j x_j^2 = p_2, \qquad U_2 = \sum_j x_j^2(x_j^2 - 1) = p_4 - p_2,$$

$$U_3 = \sum_j x_j^2(x_j^2 - 1)(x_j^2 - 4) = p_6 - 5p_4 + 4p_2, \quad \dots$$

Polynomiality theorem 2 (The Strict Case)

Definition

For each box u in $S(\lambda)$, we define its content c_u by

0	1	2	3	4	5	6
	0	1	2	3	4	
		0	1			

Note that c_u are always non-negative integers.

Polynomiality theorem 2 (The Strict Case)

Polynomiality Theorem 2 (Han-Xiong (2016), [M])

For any symmetric function $f \in \Lambda$,

$$\sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} f(\widehat{c}_u \ : \ u \in S(\lambda))$$

is a polynomial in n. Here, we set

$$\widehat{c}_u = \frac{1}{2} c_u(c_u + 1).$$

Why do we consider $\widehat{c}_u$ instead of c_u in the strict case?

- The Ordinary Case: a content c_u is an eigenvalue of Jucys-Murphy elements $J_k = (1,k) + (2,k) + \cdots + (k-1,k) \in \mathbb{Q}[\mathfrak{S}_n]$.
- The Strict Case: the quantity $\widehat{c}_u$ is the square of an eigenvalue of *spin* Jucys-Murphy elements [Vershik-Sergeev (2008)].

Our proof of Polynomiality Theorem 2

Key proposition ([M])

The algebra Γ coincides with the algebra generated by the functions on $\mathcal{SP}$

- $\lambda \mapsto |\lambda|$;
- $\lambda \mapsto f(\widehat{c}_u \ : \ u \in S(\lambda)) \ \ (f \in \Lambda)$.

Remark: The counterpart in The Ordinary Case is proved in [Olshanski (2010)].

Example

$$p_2(\widehat{c}_u \ : \ u \in S(\lambda)) = \sum_{u \in S(\lambda)} \left(\frac{c_u(c_u+1)}{2} \right)^2$$
$$= \frac{1}{20} p_5(\lambda) - \frac{1}{12} p_3(\lambda) + \frac{1}{30} p_1(\lambda).$$

A formula for content evaluations

A strict version of the theorem of Fujii-Kanno-Moriyama-Okada.

Theorem (Han-Xiong (2016))

For each $r \geq 1$, define a symmetric function V_r by

$$V_r(x_1, x_2, \dots) = \sum_{j \geq 1} \prod_{i=0}^{r-1} (x_j - \frac{i(i+1)}{2}).$$

Then

$$\sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)} (g^\lambda)^2}{n!} V_r(\widehat{c}_u \ : \ u \in S(\lambda)) = \frac{(2r)!}{(r+1)!} \binom{n}{r+1}$$

$$V_1 = p_1, \quad V_2 = p_2 - p_1, \quad V_3 = p_3 - 4p_2 + 3p_1, \dots$$

Stanley's polynomiality theorem 3 (The Ordinary Case)

Definition

Hook-length h_u for each $u \in Y(\lambda)$

7	5	2	1
4	2		
3	1		
1			

Theorem (Stanley (2010))

For any symmetric function $f \in \Lambda$,

$$\sum_{\lambda \in \mathcal{P}_n} \frac{(f^\lambda)^2}{n!} f(h_u^2 \ : \ u \in Y(\lambda))$$

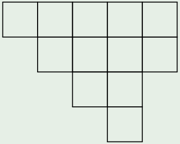
is a polynomial in n.

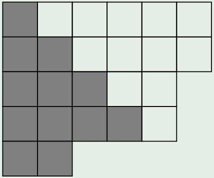
The Strict Case hook-lengths

Let λ be a strict partition.
Recall the shifted Young diagram $S(\lambda)$ and the double diagram $D(\lambda)$.

Example (Diagrams and hook-lengths h_u)

$\lambda = (5, 4, 2, 1) \in \mathcal{SP}_{12}$.

$S(\lambda) =$ 

$D(\lambda) =$ 

10	9	7	6	5	2
9	8	6	5	4	1
7	6	4	3	2	
6	5	3	2	1	
2	1				

Let $f \in \Lambda$. Consider two kinds of functions f^{HS} and f^{HD} on $\mathcal{SP}$ defined by

$$f^{\mathrm{HS}}(\lambda) = f(h_u^2 \ : \ u \in S(\lambda)),$$
$$f^{\mathrm{HD}}(\lambda) = f(h_u^2 \ : \ u \in D(\lambda)).$$

Polynomiality Theorem 3 (The Strict Case)

$$f^{\mathrm{HS}}(\lambda) = f(h_u^2 \ : \ u \in S(\lambda)), \qquad f^{\mathrm{HD}}(\lambda) = f(h_u^2 \ : \ u \in D(\lambda)).$$

Theorem ([M])

For power-sums $f = p_k$, we see that $f^{\mathrm{HS}} \in \Lambda \setminus \Gamma$ and $f^{\mathrm{HD}} \in \Gamma$.

Corollary (Han-Xiong (2016b), [M])

For any symmetric function $f \in \Lambda$,

$$\sum_{\lambda \in \mathcal{SP}_n} \frac{2^{n-\ell(\lambda)}(g^\lambda)^2}{n!} f(h_u^2 \ : \ u \in D(\lambda))$$

is a polynomial in n. (However, if we replace $D(\lambda)$ by $S(\lambda)$, then the average is not a polynomial in n.)

A formula for hook-length evaluations

The Ordinary Case

Theorem (Panova (2012) conjectured by Okada)

For each $r \geq 0$,

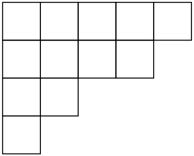
$$\sum_{\lambda \in \mathcal{P}_n} \frac{(f^\lambda)^2}{n!} \sum_{u \in Y(\lambda)} \prod_{i=1}^{r} (h_u^2 - i^2) = \frac{(2r)!}{2(r+1)!} \binom{2r+2}{r+1} \binom{n}{r+1}.$$

The Strict Case

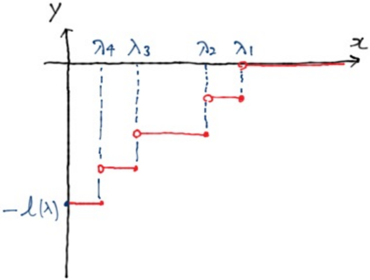
The counterpart of this identity is not found.

Settings for Limit Shape Problem

- We deal with a (ordinary) Young diagram $Y(\lambda)$ (not a shifted diagram $S(\lambda)$) of a strict partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in \mathcal{SP}_n$.

$Y(\lambda) =$ 

- The Young diagram $Y(\lambda)$ is identified with the step function $y = \lambda(x)$ $(x \geq 0)$ given by



Settings for Limit Shape Problem

- The derivative of the function $\lambda(x)$ is

$$\lambda' = \sum_{j=1}^{\ell(\lambda)} \delta_{\lambda_j}$$

- Next, we consider the scaling

$$y = \lambda^{\sqrt{n}}(x) := \frac{1}{\sqrt{n}} \lambda(\sqrt{n} x).$$

- Note that

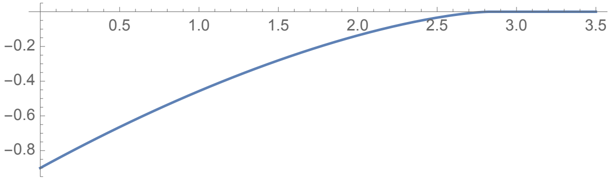
$$\int_0^\infty \lambda(x)\, dx = -n, \qquad \int_0^\infty \lambda^{\sqrt{n}}(x)\, dx = -1.$$

Limit Shape Problem

Consider a sequence $(\lambda_{(n)})_{n \geq 1}$ of random strict partitions $\lambda = \lambda_{(n)}$ of n. We want to show that, in the limit $n \to \infty$, the random variables $\lambda^{\sqrt{n}}$ converge (in some sense) to a function Ω.

Limit Shape Problem (The Strict Case): Conjecture

A candidate of Limit Shape
[Ivanov (2006), Bernstein-Henke-Regev (2007)].
English style, unshifted Young diagram $Y(\lambda)$



$$y = \Omega_{\mathrm{strict}}(x) = \begin{cases} \frac{1}{\pi}\left(x \arccos \frac{x}{2\sqrt{2}} - \sqrt{8 - x^2}\right) & (0 \le x \le 2\sqrt{2}) \\ 0 & (x > 2\sqrt{2}). \end{cases}$$

$\Omega'_{\mathrm{strict}}(x) = \frac{1}{\pi} \arccos \frac{x}{2\sqrt{2}}$ $(0 \le x \le 2\sqrt{2})$,
$\Omega_{\mathrm{strict}}(0) = -\frac{2\sqrt{2}}{\pi} = -0.90\ldots$

Law of Large Numbers

Conjecture (1st form, uniform convergence)

Let $\lambda_{(n)}$ be a random strict partition of size n distributed according to $\mathbb{P}_n^{\mathrm{SPl}}$. Then, in probability,

$$\lim_{n\to\infty} \sup_{x>0} \left| (\lambda_{(n)})^{\sqrt{n}}(x) - \Omega_{\mathrm{strict}}(x) \right| = 0.$$

Conjecture (2nd form, weak convergence)

Let $\lambda_{(n)}$ be a random strict partition of size n distributed according to $\mathbb{P}_n^{\mathrm{SPl}}$. Then, in probability,

$$\lim_{n\to\infty} \int_0^\infty x^k (\lambda_{(n)})^{\sqrt{n}}(x) dx = \int_0^\infty x^k \Omega_{\mathrm{strict}}(x)\, dx$$

for any $k = 0, 1, 2, \ldots$.

One can compute the integral $\int_0^\infty x^k \Omega_{\mathrm{strict}}(x)\, dx$ easily.

Law of Large Numbers

Conjecture (2nd form) is equivalent to the 3rd form.

Conjecture (3rd form)

Let $\lambda_{(n)}$ be as before. Then, in probability,

$$\lim_{n\to\infty} \frac{p_k(\lambda_{(n)})}{n^{\frac{k+1}{2}}} = \frac{2^{\frac{3(k+1)}{2}}\Gamma(1+\frac{k}{2})}{\sqrt{\pi}(k+1)^2\Gamma(\frac{k+1}{2})} \qquad \text{for any } k = 1, 2, \dots.$$

Conjecture (4th form, much weaker version)

$$\lim_{n\to\infty} \frac{\mathbb{E}_n^{\mathrm{SPl}}[p_k]}{n^{\frac{k+1}{2}}} = \frac{2^{\frac{3(k+1)}{2}}\Gamma(1+\frac{k}{2})}{\sqrt{\pi}(k+1)^2\Gamma(\frac{k+1}{2})} \qquad \text{for any } k = 1, 2, \dots.$$

Law of Large Numbers

Conjecture (4th form, much weak version)

$$\lim_{n\to\infty} \frac{\mathbb{E}_n^{\mathrm{SPl}}[p_k]}{n^{\frac{k+1}{2}}} = \frac{2^{\frac{3(k+1)}{2}}\Gamma(1+\frac{k}{2})}{\sqrt{\pi}(k+1)^2\Gamma(\frac{k+1}{2})} \qquad \text{for any } k = 1, 2, \dots.$$

Recall a Matsumoto-Okada identity

$$\mathbb{E}_n^{\mathrm{SPl}}[\varphi_k] = \frac{2^k(2k+1)!}{((k+1)!)^2} \cdot n(n-1)(n-2)\cdots(n-k)$$

for $\varphi_k = \sum_{i\ge1}\prod_{j=-k}^{k}(x_i+j)$. This implies that $\mathbb{E}_n^{\mathrm{SPl}}[p_{2k+1}]$ is a polynomial in n of degree $k+1$ and

$$\frac{\mathbb{E}_n^{\mathrm{SPl}}[p_{2k+1}]}{n^{k+1}} \to \frac{2^k(2k+1)!}{((k+1)!)^2} = \left[\frac{2^{\frac{3(k+1)}{2}}\Gamma(1+\frac{k}{2})}{\sqrt{\pi}(k+1)^2\Gamma(\frac{k+1}{2})}\right]_{k\mapsto 2k+1}$$

Therefore Conjecture (4th form) holds true for k odd.

Law of Large Numbers

Conjecture (4th form, weak version)

In the limit $n \to \infty$,

$$\mathbb{E}_n^{\mathrm{SPl}}[p_k] \sim \frac{2^{\frac{3(k+1)}{2}} \Gamma(1+\frac{k}{2})}{\sqrt{\pi}(k+1)^2 \Gamma(\frac{k+1}{2})} \cdot n^{\frac{k+1}{2}} \qquad \text{for any } k = 1, 2, \dots .$$

- However, if k is even, then $\mathbb{E}_n^{\mathrm{SPl}}[p_k]$ is not a polynomial in n.
- So, our Main Polynomiality Theorem (Kerov' approach, seen in [Ivanov-Olshanski (2001)]) does not work for k even. (This is a difficulty of The Strict Case, unlike The Ordinary Case.)

How can one prove Conjecture (4th form) for k even?

Our Strategy

Conjecture (4th form, weak version)

In the limit $n \to \infty$,

$$\mathbb{E}_n^{\mathrm{SPl}}[p_k] \sim \frac{2^{\frac{3(k+1)}{2}}\Gamma(1+\frac{k}{2})}{\sqrt{\pi}(k+1)^2\Gamma(\frac{k+1}{2})} \cdot n^{\frac{k+1}{2}} \qquad \text{for } k \text{ even.}$$

Joint work with Tomoyuki Shirai (Kyushu University), in progress

- A random strict partition defines a Pfaffian point process on $\mathbb{Z}$ ([M (2005)]).
- The 1-point correlation function is explicitly given by using Bessel functions.
- The expectation $\mathbb{E}_n^{\mathrm{SPl}}[p_k]$ is expressed as a computable infinite sum involving Bessel functions.
- We have a heuristic proof of Conjecture (4th form) but it is not rigorous yet.

Other open problems in The Strict Case

The Ordinary Case

Normalized character: for $\mu \in \mathcal{P}_k$ and $\lambda \in \mathcal{P}_n$ with $n \geq k$,

$$\mathrm{Ch}_\mu(\lambda) = n(n-1)\cdots(n-k+1)\frac{\chi^\lambda_{\mu\cup(1^{n-k})}}{f^\lambda}.$$

- the function Ch_μ is a shifted-symmetric function.
- Kerov polynomial: Ch_μ is a polynomial in free cumulants with nonnegative integer-valued coefficients.
- Stanley-Féray-Śniady polynomial: Ch_μ is a polynomial in multi-rectangular coordinates of Young diagrams.

We can define a strict (or spin or projective) version $\mathrm{Ch}^{\mathrm{strict}}_\rho$ for odd partition ρ. The function $\mathrm{Ch}^{\mathrm{strict}}_\rho$ on $\mathcal{SP}$ belongs to Γ.

Open problems

Find analogues of Kerov polynomials and S-F-Ś polynomials.

Thank you for listening!

Merci de votre attention! ご静聴ありがとうございました。

1. Introduction: two kinds of Plancherel measures.
2. Polynomiality of Plancherel Averages
   - Main Polynomiality Theorem
   - Proof of Polynomiality Theorem. Factorial Schur Q-functions
   - Degree estimate
3. Content Evaluations
4. Hook-Length Evaluations
5. Limit Shape Problem (joint work with Tomoyuki Shirai, in progress)