Harmonic functions on multiplicative graphs and weight polytopes of representations

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- Study minimal harmonic functions on rooted graded graphs generalizing Young lattice.
- Study the conditionning of random paths to stay in Weyl chambers.

I. Simple random walk

Let $B = \{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and let $\overline{C}$ be the cone

$$\overline{C} = \{x \in \mathbb{R}^n \mid x_1 \geq \cdots \geq x_n \geq 0\} \subset \mathbb{R}^n.$$

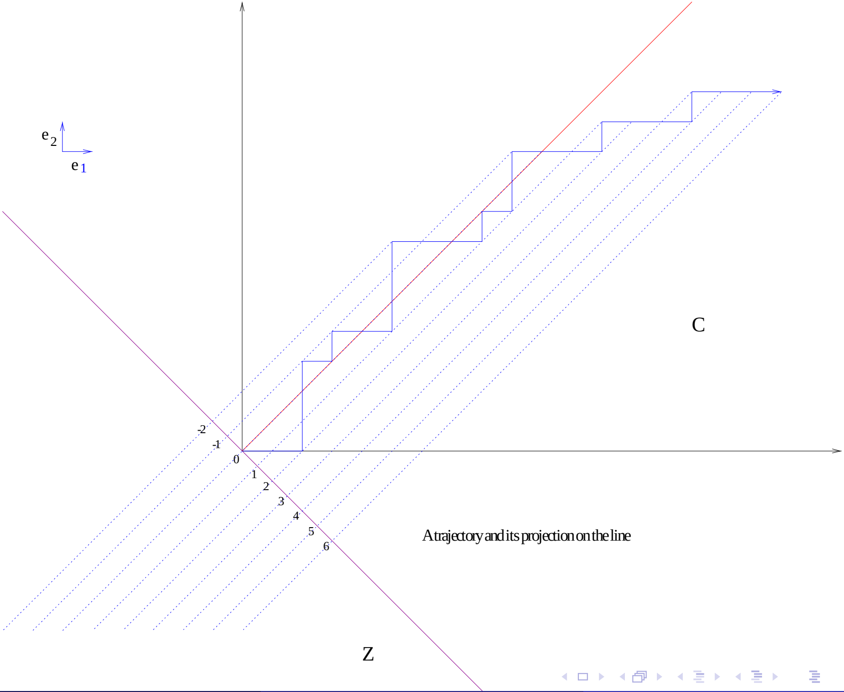
The elements of $P_+ = \overline{C} \cap \mathbb{Z}^n$ are partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$.

Let $(X_\ell)_{\ell \geq 1}$ be a sequence of random variables in B (i.i.d.)

$$\mathbb{P}(X_\ell = e_i) = p_{e_i} \in [0,1] \text{ for } i = 1, \ldots, n$$
$$p_{e_1} + \cdots + p_{e_n} = 1$$

$$m := E(X_\ell) = \sum_{i=1}^{n} p_{e_i} e_i.$$

$S_\ell = X_1 + \cdots + X_\ell$ defines a random walk on $\mathbb{Z}^n$ with steps in B.



A trajectory and its projection on the line

For any $\beta \in \mathbb{Z}^n$, set $p^\beta = p_{e_1}^{\beta_1} \cdots p_{e_n}^{\beta_n}$. For $\mu \in P_+$, consider

$$\psi(\mu) = p^{-\mu}\mathbb{P}(S_0 = \mu, S_\ell \in \overline{C}, \forall \ell \geq 1).$$

Lemma

1. *The function ψ is nonnegative on $P_+ = \overline{C} \cap \mathbb{Z}^n$.*

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Lemma

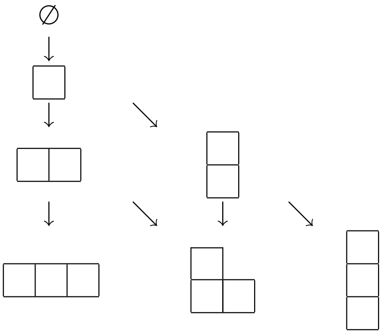
1. *The function ψ is nonnegative on $P_+ = \overline{C} \cap \mathbb{Z}^n$.*
2. *The function is harmonic on P_+*

$$\psi(\mu) = \sum_{\lambda/\mu = \square} \psi(\lambda)$$

where the sum is over the partitions $\lambda \supset \mu$ such that $|\lambda| - |\mu| = 1$.

Harmonic functions on the Young lattice

Let $\mathcal{Y}_n$ be the Young graph of partitions with at most n parts.



Definition

A function h is positive harmonic on $\mathcal{Y}_n$ when $h(\lambda) > 0$ for any λ and

$$h(\mu) = \sum_{\lambda/\mu=\square} h(\lambda).$$

The set $\mathcal{H}_{\mathcal{Y}_n}$ of positive harmonic functions on $\mathcal{Y}_n$ with $h(\emptyset) = 1$ is convex.

$$h_1, h_2 \in \mathcal{H}_{\mathcal{Y}_n} \Longrightarrow ah_1 + (1-a)h_2 \in \mathcal{H}_{\mathcal{Y}_n} \quad \forall a \in [0,1].$$

Let $\mathcal{E}_{\mathcal{Y}_n}$ be its subset of minimal functions.

Theorem (O'Connell (2004))

Assume $m = (p_{e_1} \geq \cdots \geq p_{e_n}) \in \overline{C}$. *For any partition* λ,

$$\psi(\lambda) = \prod_{1 \leqslant i < j \leqslant n} \left(1 - \frac{p_{e_j}}{p_{e_i}}\right) s_\lambda(p_{e_1}, \ldots, p_{e_n}) \text{ thus,}$$

$$\mathbb{P}_\mu(S_\ell \in \overline{C}, \forall \ell \geq 1) = p^{-\lambda}\psi(\lambda)$$

where s_λ *is the Schur polynomial associated to* λ.

Extremal harmonic function

Recall that $\mathcal{E}_{\mathcal{Y}_n}$ is the set of extremal positive harmonic functions f on $\mathcal{Y}_n$ s.t. $f(\emptyset) = 1$.
Let

$$T_n = \{(p_1 \geq \cdots \geq p_n \geq 0 \mid p_1 + \cdots + p_n = 1\} = \mathrm{conv}(B) \cap \overline{C}.$$

Theorem (f.d. version of Thoma's simplex)

The map

$$\theta : \begin{cases} T_n \to \mathcal{E}_{\mathcal{Y}_n} \\ (p_1, \dots, p_n) \longmapsto f : \begin{cases} P_+ \to \mathbb{R}_{>0} \\ \lambda \longmapsto s_\lambda(p_1, \dots, p_n) \end{cases} \end{cases}$$

is a bijection.

II Generalizations

Problem

Introduce random trajectories with steps the weights of a f.d. irreducible representation V of any simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and study their probability to remain in the Weyl chamber.

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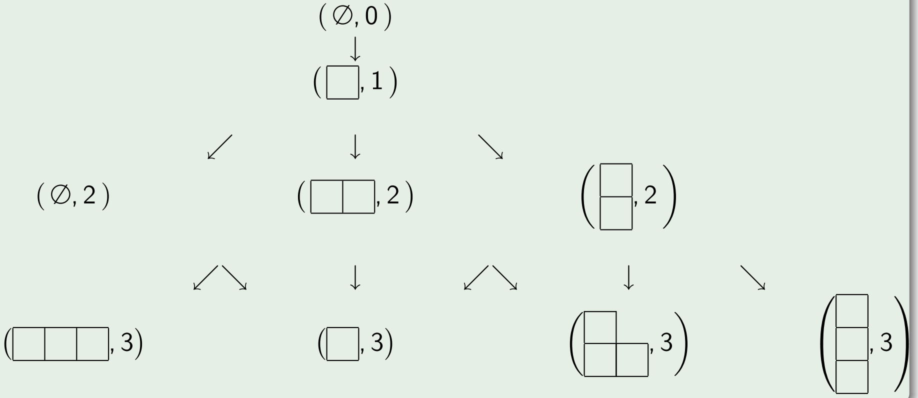
Find extremal positive harmonic functions on the graded rooted graph Γ_V with

- *root* $(0,0)$,
- *vertices: the* $(\lambda,\ell)\in P_+\times\mathbb{Z}_{\geq 0}$ *s.t.* $V(\lambda)$ *is an irreducible component of* $V^{\otimes \ell}$,
- *arrows:* $(\mu,\ell-1)\overset{c^{\lambda}_{\mu,V}}{\rightarrow}(\lambda,\ell)$ *where* $c^{\lambda}_{\mu,V}=[V(\lambda):V(\mu)\otimes V]$.

Example

Part of Γ_V for V the defining rep. of $\mathfrak{sp}_{2n}$ with set of weights $\pm e_i$ in $\mathbb{Z}^n$.



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- $\rightarrow$ only particular continuous trajectories (Littelmann paths) have a nice general algebraic interpretation in representation theory
- $\rightarrow$ so we need to replace random walks by random (continuous) trajectories obtained as the concatenation of Littelmann paths in the crystal of V.

Littelmann path model

Let $\mathfrak{g}$ be a simple Lie algebra with root system R, simple roots $\alpha_1, \dots, \alpha_n$, weight lattice P and Weyl chamber $\overline{C}$.

- A Littelmann path is a piecewise linear map $\eta : [0,1] \to P_{\mathbb{R}}$ with rational turning points and such that $\eta(0) = 0$ and $\eta(1) \in P$.
- Crystal operators $\tilde{f}_i$, $i = 1, \dots, n$ act on Littelmann paths η by reflecting some parts of η by $s_{\langle \alpha_i \rangle^\perp}$.
- A highest weight path η is such that $\mathrm{Im}\, \eta \subset \overline{C}$.

Crystal of an irreducible module

Consider $\kappa \in P_+$

- Choose η_κ a h.w.p such that $\eta(1) = \kappa$. The set

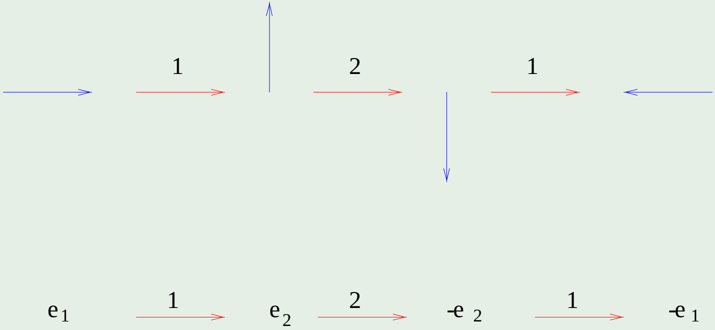
  $$B(\eta_\kappa) = \{\tilde{F} \cdot \eta_\kappa \mid \tilde{F} \text{ product of } \tilde{f}_i\}$$

  is the crystal associated to η_κ.
- The set $\Pi(\kappa) = \{\eta(1) \mid \eta \in B(\eta_\kappa)\}$ is the set of weights of the irreducible f.d. $\mathfrak{g}$-mod $V(\kappa)$.

Example

In type C_2, $P = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \mathbb{R}^2$, $\overline{C} = \{x = (x_1, x_2) \mid x_1 \geq x_2 \geq 0\}$ and $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = 2\varepsilon_2$.
For $\kappa = \omega_1 = e_1$,

Crystal of the vector representation in type C2

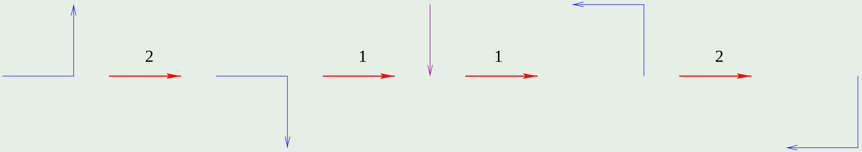


The Littelmann paths are lines (as in any minuscule representation)

Example

For $\kappa = \omega_2 = e_1 + e_2$,

In type C2, the crystal of the fundamental representation with dimension 5 with its 5 elementary Littelman paths



is the up-down path of weight 0

Random trajectory

Assume $B(\eta_\kappa)$ has probability distribution $p = (p_\eta)_{\eta \in B(\eta_\kappa)}$

$$\sum_{\eta \in B(\eta_\kappa)} p_\eta = 1.$$

Set

$$\mathbf{m} := \sum_{\eta \in B(\eta_\kappa)} p_\eta \eta$$

and $\mathbf{m}(1) = m$.

Definition

- For any $\ell \geq 1$ let $\mathcal{W}_\ell$ be the concatenation of ℓ elementary paths randomly choosen according to the distribution p. It has length ℓ.
- $\mathcal{W} := (\mathcal{W}_\ell)_{\ell \geq 1}$ is a random trajectory.

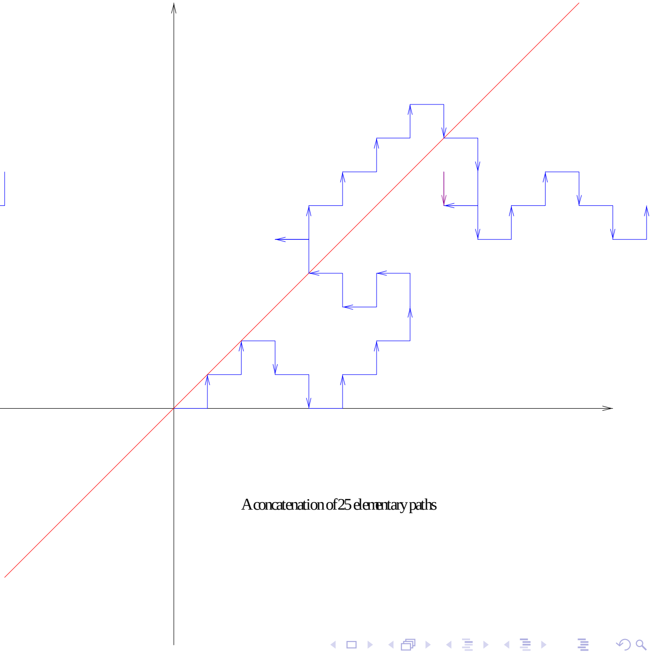
Set $W_\ell = \mathcal{W}(\ell)$.

The sequence $W = (W_\ell)_{\ell \geq 1}$ is a random walk with steps the weights of $V(\kappa)$.

In type C2 the 5 elementary Littelman paths

are the vertices of the crystal of the fun. rep. of dim 5

is the up-down path



A concatenation of 25 elementary paths

Central measures on trajectories

A random trajectory η of length ℓ is the concatenation

$$\eta = \pi_1 * \cdots * \pi_\ell \in B(\eta_\kappa)^{*\ell}$$

of ℓ elementary paths in $B(\eta_\kappa)$.

It has probability

$$p_\eta = p_{\pi_1} \times \cdots \times p_{\pi_\ell}.$$

Definition

The distribution p on $B(\eta_\kappa)$ is central (or harmonic) when for any $\ell \geq 1$ and η, η' in $B(\eta_\kappa)^{*\ell}$ such that $\eta(\ell) = \eta'(\ell)$, we have $p_\eta = p_{\eta'}$.

Theorem (L., Tarrago 2016)

The distribution p with $p_\eta \geq 0$ for any $\eta \in B(\eta_\kappa)$ is central iff there exists $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}_{\geq 0}$ such that

$$p_{\eta'} = p_\eta \times \tau_i$$

as soon as $\eta \xrightarrow{i} \eta'$ in $B(\eta_\kappa)$

Example

In type C_2 with $\kappa = \omega_1$, choose $\tau = (\tau_1, \tau_2) \in \mathbb{R}^2_{>0}$

$$e_1 \underset{\times\tau_1}{\overset{1}{\rightarrow}} e_2 \underset{\times\tau_2}{\overset{2}{\rightarrow}} -e_2 \underset{\times\tau_1}{\overset{1}{\rightarrow}} -e_1$$

$$p_{e_1} = \frac{1}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}, \; p_{e_2} = \frac{\tau_1}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}$$

$$p_{-e_2} = \frac{\tau_1\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}, \; p_{-e_1} = \frac{\tau_1^2\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}$$

and

$$m = \frac{1-\tau_1^2\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2} e_1 + \frac{\tau_1-\tau_1\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2} e_2$$

Observe that $m \in C$ iff $(\tau_1, \tau_2) \in]0,1[^2$.

Generalization of O'Connell's results

Assume $\boldsymbol{\tau} \in]0,1[^n$ (this is equivalent to $m \in C$).
For any $\beta = a_1\alpha_1 + \cdots + a_n\alpha_n \in P$, set $\boldsymbol{\tau}^\beta = \tau_1^{-a_1} \cdots \tau_n^{-a_n}$ (here $\beta \in \mathbb{Q}^n$).

Theorem (L., Lesigne, Peigné 2012)

We have

$$\mathbb{P}_\mu(\mathcal{W}(t) \in \overline{C} \text{ for any } t \geq 0) = \prod_{\alpha \in R_+} (1 - \boldsymbol{\tau}^\alpha)\boldsymbol{\tau}^{-\mu} s_\mu(\boldsymbol{\tau})$$

where s_μ is the Weyl character of $V(\mu)$.

Asymptotic of tensor multiplicities

Set

$$V(\mu) \otimes V(\kappa)^{\otimes \ell} = \bigoplus_{\lambda \in P_+} V(\lambda)^{\oplus f^{\ell}_{\lambda/\mu}}.$$

Assume $m \in C$ and consider $\lambda^{(\ell)}$ a sequence in P_+ such that

$$\lambda^{(\ell)} = \ell m + o(\ell) \in C.$$

Theorem (L., Lesigne, Peigné 2014)

We have

$$\lim_{\ell \to +\infty} \frac{f^{\ell}_{\lambda^{(\ell)}/\mu}}{f^{\ell}_{\lambda^{(\ell)}}} = s_\mu(\boldsymbol{\tau}).$$

IV Positive Harmonic functions on our multiplicative graphs

The function h is harmonic on $\Gamma_{V(\kappa)}$ when

$$h(\mu, \ell - 1) = \sum_{(\mu,\ell-1) \overset{c_{\mu,\kappa}^{\lambda}}{\rightarrow} (\lambda,\ell)} c_{\mu,\kappa}^{\lambda} h(\lambda, \ell)$$

Let $\mathcal{E}_\delta$ for the set of extremal positive harmonic functions f on $\Gamma_{V(\kappa)}$ such that $f(\emptyset, 0) = 1$.

Let $\Pi(\delta)$ be the set of weights of $V(\delta)$.

Theorem (L, Tarrago (2016))

For any dominant weight κ there exists a subset $[0,1]^n_\kappa \subset [0,1]^n$ such that

1. *The map $m : [0,1]^n_\kappa \to \mathrm{conv}(\Pi(\delta)) \cap \overline{C}$ s.t.*

$$m(\boldsymbol{\tau}) = \frac{1}{s_\kappa(\mathbf{t})} \sum_{\gamma \in \Pi(\delta)} \gamma \mathbf{t}^\gamma$$

is an homeomorphism.

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   is an homeomorphism.

2. *The map $\theta : [0,1]^n_\kappa \to \mathcal{E}_\kappa$ s.t.*

$$\theta(\boldsymbol{\tau}) = h : (\lambda, \ell) \mapsto \frac{s_\lambda(\tau_1, \ldots, \tau_n)}{s_\kappa(\tau_1, \ldots, \tau_n)^\ell}$$

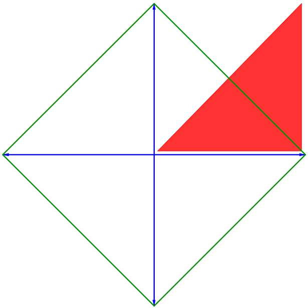
   is a bijection.

Example

For $\kappa = \omega_1$ in type C_2

$$m(\boldsymbol{\tau}) = \frac{1-\tau_1^2\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}e_1 + \frac{\tau_1-\tau_1\tau_2}{1+\tau_1+\tau_1\tau_2+\tau_1^2\tau_2}e_2$$

gives a bijection between $[0,1]^2_{\omega_1} =]0,1[^2 \bigsqcup \{1\} \times [0,1] \bigsqcup [0,1] \times \{1\}$ and $\mathrm{conv}(\pm e_1, \pm e_2) \cap \overline{C}$.



The Weyl chamber (red), the weights (blue) and the weight polytope (green)