# Spherically Symmetric Random Permutations

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# De Finetti's and Freedman's theorems revisited

**De Finetti's (30's) Thm:** An infinite exchangeable sequence $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots) \in \{0,1\}^\infty$ is a (unique) mixture of Bernoulli($p$) sequences.

Exchangeability $\leftrightarrow$ invariance of the distribution $P$ of $\boldsymbol{\xi}$ under the group $S_\infty$ of finite permutations $\leftrightarrow$ each marginal distribution $P_n, n = 1, 2, \dots,$ is invariant under $S_n$.
The theorem is equivalent to Hausdorff's characterisation of moment sequences for probability measures on $[0,1]$.

**Freedman's (60's) Thm:** An infinite spherically symmetric (ISS) sequence $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots) \in \mathbb{R}^\infty$ is a (unique) mixture of iid $\mathcal{N}(0, \sigma^2)$-sequences.

Infinite spherical symmetry $\leftrightarrow$ invariance of the distribution $P$ of $\boldsymbol{\xi}$ under the group $O_\infty$ of finite-dimensional orthogonal transformations.
The theorem is equivalent to Schoenberg's characterisation of functions $\phi$, s.t. $\phi(x_1^2 + \dots + x_n^2)$ is positive definite for every $n$.

Let $\rho_n = \|(\xi_1, \dots, \xi_n)\|$, and let $U_{n,r}$ be the uniform distribution on $n$-dim sphere with radius $r$.

- ISS $\leftrightarrow$ the conditional distribution $(\xi_1, \dots, \xi_n) \,|\, \rho_n = r$ is $U_{n,r}$ for all $n, r$.
- Every ISS probability measure $P$ is a unique convex mixture of extreme ISS measures.
- A sequence $r_\nu$ $(\nu = 1, 2, \dots)$ is called *regular* if $U_{\nu,r_\nu}$'s converge to some $P$ (in the sense of weak convergence of $n$-dim distributions). Every extreme ISS $P$ can be *induced* through a regular sequence.
- If $P$ is extreme then $U_{\nu,\rho_\nu}$ converge to $P$ almost surely, that is $\rho_\nu$ is regular almost surely.
- Under $\otimes\mathcal{N}(0, \sigma^2)$, a LLN holds: $\rho_\nu \sim \sigma\nu^{1/2}$ a.s.
- The family $\otimes\mathcal{N}(0, \sigma^2)$ depends continuously on $\sigma$.

- *Propagation of stochastic monotonicity*: if $P_\nu, P'_\nu$ are spherically symmetric in $\mathbb{R}^\nu$ and such that $\rho_\nu$ under $P_\nu$ is stochastically greater than under $P'_\nu$ (i.e. $E_P[g(\rho_\nu)] \geq E_{P'} g(\rho_\nu)$ for every increasing $g$), then the same is true for every $\rho_n, \; n \leq \nu$.

Equivalently: for a random point sampled uniformly from a sphere, the bigger the sphere the bigger (stochastically) the projection.

By a sandwich argument using the stochastic monotonicity:

- a sequence is regular iff $r_\nu \sim \sigma \nu^{1/2}$ for some $\sigma \in [0, \infty)$, in which case the induced $P$ is $\otimes \mathcal{N}(0, \sigma^2)$.

Freedman's Thm follows by noting that all measures $\otimes \mathcal{N}(0, \sigma^2)$'s are disjoint for distinct $\sigma$.

Replacing the Euclidean distance in $\mathbb{R}^n$ by the $L^p$-distance leads to mixtures of iid sequences with density $c(\lambda) e^{-\lambda |x|^p}$ (Berman '80).

To derive de Finetti's Thm using this path, $\|(\xi_1, \dots, \xi_n)\|$ should be seen as the Hamming distance to $(0, \dots, 0)$, and the regularity condition becomes $r_\nu \sim p\,\nu$.

Applications of stochastic monotonicity to the boundary problem on graded graphs appear in AG–Pitman (Stirling triangles), and Bufetov–VG (Young lattice).

# Spaces of virtual permutations

Let $f_n : S_n \to S_{n-1}$ be $n$-to-1 projections ($n > 1$).
If $f_n(\pi) = \sigma$, we say that $\pi$ is an extension of $\sigma$, and that $\sigma$ is the projection of $\pi$.
Consider $S^\infty$, the projective limit of $(S_n, f_n)$'s. Each $\boldsymbol{\pi} \in S^\infty$ is a (generalised) *virtual permutation*.
A random virtual permutation is a probability measure $P$ on $S^\infty$, uniquely determined by consistent marginal distributions, $P_{n-1} = f_n(P_n)$.

Cardinal example: the *uniform* virtual permutation has distribution $P^*$ such that each marginal distribution $P_n^*$ uniform on $S_n$.

- Endow each $S_n$ with a metric, and let $\|\pi_n\|$ be the distance to the identity in $S_n$. A probability measure $P$ on $S^\infty$ is called *infinitely spherically symmetric* if for every $n$ the conditional distribution of $\pi_n$ given $\|\pi_n\| = r$ is $U_{n,r}$, that is uniform on the sphere in $S_n$ of radius $r$.

To ensure that spherical symmetry is preserved under projections, the metric should be *compatible* with projections in the following sense: $n = 1, 2, \ldots$

- the number of extensions $\#\{\pi \in S_n : \|\pi\| = r, f_n(\pi) = \sigma\}$ only depends on $\|\sigma\|$,

which implies that $f_n(U_{n,r})$ is a mixture of $U_{n-1,\bullet}$'s.

- A ISS virtual permutation $P$ corresponds uniquely to a Markov chain $\|\pi_1\|, \|\pi_2\|, \ldots$, which has *backward* transition probabilities $\|\pi_{n-1}\| \leftarrow \|\pi_n\|$ same as under the uniform $P^*$.
- By the virtue of Doob's $h$-transform this can be recast in terms of harmonic functions for the Markov chain $\|\pi_1\|, \|\pi_2\|, \ldots$ under $P^*$.

Metrics on $S_n$: Chritchlow ('85), Diaconis ('88)... Which are consistent with *some* projections?

There is a natural choice of projections for Hamming, Cayley and Kendall-tau metrics.

# Hamming spherical symmetry

For the Hamming distance on $S_n$, $\|\pi\| = n - F(\pi) \in \{0, 2, 3, \ldots, n\}$, where

$$F(\pi) = \#\{j \in [n] : \pi(j) = j\}, \quad \pi \in S_n$$

is the number of fixed points. A $\pi \in S_n$ with $\|\pi\| = n$ is a *derangement*. A projection $f_n : S_n \to S_{n-1}$ compatible with the $F$-statistic is the operation of deleting $n$ from the cycle notation. For instance, $(13)(24) \in S_4$ has five extensions

$$(153)(24), (135)(24), (13)(254), (13)(245), (13)(24)(5).$$

With such $f_n$'s, $S^\infty$ is the space of virtual permutations introduced by Kerov-Olshanski-Vershik.

- Under ISS distribution $P$, $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots)$ has each $\pi_n$ conditionally uniform given the number of fixed points $F(\pi_n)$.

# Permutations with many fixed points

**Definition** Fix $\alpha \in (0,1]$ and apply the following rules to define a virtual permutation $P^\alpha$

- each $n$ independently of other elements is *singleton* with probability $\alpha$, and non-singleton with probability $1-\alpha$,
- every singleton $n$ is a fixed point in $\pi_\nu$ for $\nu \geq n$,
- the virtual permutation restricted to the set of nonsingleton elements is distributed like under $P^*$, provided the nonsingleton elements are enumerated in increasing order by $\mathbb{N}$.

For $\alpha = 0$ set $P^0 = P^*$.

**Sequential construction of** $P^{\alpha}$ (extended Dubins-Pitman *Chinese restaurant process*):
Given $\pi_n$ with some $k$ singletons, element $n+1$ is singleton with probability $\alpha$, and with the same probability $(1-\alpha)/(n-k+1)$ the element is inserted in a cycle clockwise next to any of $n-k$ nonsingleton elements, or appended to $\pi_n$ as a cycle $(n+1)$.

Under $P^{\alpha}$

- the probability of any permutation with $k$ fixed points is

$$p_{n,k} = \sum_{j=0}^{k} \binom{k}{j} \alpha^j (1-\alpha)^{n-j} \frac{1}{(n-j)!}$$

- $F(\pi_n)/n \to \alpha$ almost surely.

**Paintbox construction of $P^\alpha$:** Split $[0,1]$ at point $1-\alpha$, then split $[0,1-\alpha]$ in subintervals by the iterated uniform stick-breaking. Sample $X_1,\dots,X_n$ iid from Uniform$[0,1]$. If $X_j > 1-\alpha$, integer $j$ is a fixed point of $\pi_n$. Otherwise, integers $i,j$ belong to the same cycle of $\pi_n$ iff $X_i, X_j$ belong to the same partition interval within $[0,1-\alpha]$.

**Theorem** Every Hamming-ISS virtual permutation $P$ is a unique mixture of $P^\alpha, \alpha \in [0,1]$.

A proof employs the propagation of stochastic monotonicity property and the law of large numbers.

**Example** The uniform[0,1]-mixture over $\alpha$ yields

$$p_{n,k} = \frac{k!}{(n+1)!} \sum_{j=0}^{k} \frac{1}{j!}$$

(probability of any permutation with $F(\pi_n) = k$).

# Enumeration of Hamming spheres in $S_n$

The number of derangements in $S_n$ is

$$d_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

and the number of permutations with $k$ fixed points is

$$D_{n,k} = \binom{n}{k} d_{n-k},$$

satisfying a Pascal-type recursion

$$D_{n,k} = (n-k-1)D_{n-1,k} + D_{n-1,k-1} + (k+1)D_{n-1,k+1},$$

(with $D_{1,0} = 0, D_{1,1} = 1$).

Every Hamming-ISS probability measure $P$ is determined by a nonnegative probability function $(p_{n,k})$ that satisfies a dual backward recursion

$$p_{n,k} = (n-k)p_{n+1,k} + p_{n+1,k+1} + kp_{n+1,k+1}, \quad k \in \{0, \dots, n-2, n\}$$

(where $p_{1,1} = 1$).
The extreme solutions can be obtained as limits of the Martin kernel

$$p_{n,k}^{\nu,\varkappa} = \frac{D_{n,k}^{\nu,\varkappa}}{D_{\nu,\varkappa}},$$

where $D_{n,k}^{\nu,\varkappa}$ enumerates the number of extensions of fixed permutation $\pi \in S_n$ with $F(\pi) = k$ to a permutation $\sigma \in S_\nu$ with $F(\sigma) = \varkappa$.

The identification of extreme Hamming-ISS permutations is equivalent to:

**Theorem** The Martin kernel converges along the sequence $\varkappa = \varkappa_\nu$ ($\leftrightarrow \varkappa_\nu$ is regular) iff $\varkappa \sim \alpha n$ for some $\alpha \in [0,1]$, in which case the limit corresponds to $P^\alpha$.

A direct proof is possible via the recursion on $p_{n,k}$ and explicit formula

$$D_{n,0}^{\nu,\varkappa} = \frac{(\nu-n)!}{\varkappa!} \sum_{m=0}^{\nu-n-\varkappa} \binom{n+m-1}{m} \frac{d_{\nu-n-m-\varkappa}}{(\nu-n-n-\varkappa)!}$$

The cycle partitions for Hamming-ISS permutation comprise *Kingman's partition structure* with 'exchangeable partition probability function' of the form

$$p(n_1,\ldots,n_\ell) = p_{n,k} \prod_{j=1}^{\ell} (n_j-1)!$$

where $k$ is the number of 1's in $n_1,\ldots,n_\ell$.

The Kendall-tau distance between $\pi, \sigma \in S_n$ is the number of discordant positions $i < j$ with

$$\mathrm{sgn}(\pi(i) - \pi(j)) = -\mathrm{sgn}(\sigma(i) - \sigma(j))$$

Then $|\pi|$ coincides with the number of inversions

$$I(\pi) = \#\{(i, j) : 1 \leq i < j \leq n, \pi(i) > \pi(j)\}.$$

This statistic is compatible with any of the following two projections $f'_n, f''_n : S_n \to S_{n-1}$. Writing $\pi \in S_n$ in one-line notation:

(i) $f'_n$ deletes the last entry $\pi(n)$, and re-labels $\pi(1), \dots, \pi(n-1)$ by an increasing bijection with $[n-1]$, e.g. $2, 5, 1, 4, 3 \to 2, 4, 1, 3$.

(ii) $f''_n$ deletes letter $n$ from the online notation of $\pi$, e.g. $2, 5, 1, 4, 3 \to 2, 1, 4, 3$.

Adopting projection $f_n'$, we encode virtual permutation $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots)$ into infinite *Lehmer code* $\boldsymbol{\pi} \to (\eta_1, \eta_2, \dots)$, where
$\eta_n = \#\{j \in [n] : \pi_n(j) > \pi_n(n)\}$
For instance, $2, 5, 1, 4, 3 \leftrightarrow 0, 0, 2, 1, 2$.
In these coordinates, $S^\infty$ is a product space, and $I(\pi_n) = \eta_1 + \dots + \eta_n$.
Under the uniform distribution $P^*$ the variables $\eta_n$ are independent with $\eta_n$ distributed uniformly on $\{0, \dots, n-1\}$.

Exponential tilting with parameter $q$ does not affect the conditional distributions $(\eta_1, \dots, \eta_n) \,|\, \eta_1 + \dots + \eta_n = k$, yields truncated geometric distribution for $\eta_n$ and the *Mallows distribution* on each $S_n$

$$P^q(\pi_n) = \frac{q^{I(\pi_n)}}{n_q!},$$

where $n_q = (1 - q^n)/(1 - q)$

- The law of large numbers: under Mallows distribution $P^q$, as $n \to \infty$

$$I(\pi_n) \sim \begin{cases} \frac{q}{1-q} n, & \text{for } 0 \leq q < 1, \\ \frac{1}{4} n^2, & \text{for } q = 1, \\ \binom{n}{2} - \frac{q^{-1}}{1-q^{-1}} n, & \text{for } 1 < q \leq \infty. \end{cases}$$

  (Bhatnagar–Peled '17 give finer asymptotics for Mallows($q$) )

- The process of sums $\eta_1 + \cdots + \eta_n$, $n = 1, 2, \ldots$ with uniform $\eta_j$'s has the property of propagation of stochastic monotonicity.

**Theorem** Every Kendall-tau-ISS virtual permutation $P$ is a mixture of Mallows distributions $P^q, q \in [0, \infty]$.

$M_{n,k}$ counts solutions to $\eta_1 + \cdots + \eta_n = k$ with $\eta_j \in \{0, \ldots, j-1\}$
$M^{\nu,\varkappa}_{n,k}$ counts solutions to $\eta_{n+1} + \cdots + \eta_\nu = \varkappa - k$ with $\eta_j \in \{0, \ldots, j-1\}$

**Corollary** The Martin kernel

$$p^{\nu,\varkappa}_{n,k} = \frac{M^{\nu,\varkappa}_{n,k}}{M_{n,k}}$$

converges as $\nu \to \infty$ if either $\varkappa \sim c\,\nu$, or $\varkappa \sim \binom{\nu}{2} - c\,\nu$ (for $0 \le c < \infty$), or $\varkappa \gg \nu$ and $\frac{\nu^2}{2} - \varkappa \gg n$. In the last case the limit is $p_{n,k} = 1/n!$.

Cayley distance on $S_n$ is the minimal number of transpositions needed to transform one permutation into another. Then $\|\pi\| = n - C(\pi)$, with $C(\pi)$ the number of cycles of $\pi \in S_n$.
For the projections we choose $f_n$ (removing $n$ from the cycle notation), so $S^\infty$ is the Kerov-Vershik-Olshanski space of virtual permutations.

Then $C(\pi_n) = \sum_{j=1}^{n} \beta_j$, where $\beta_j$ is the indicator of $\pi_j(j) = j$. Under $P^*$ the indicators are independent, with $\beta_j$ being Bernoulli$(1/j)$, and the exponential tilting yields *Ewens distributions*

$$P^\theta(\pi_n) = \frac{\theta^{C(\pi_n)}}{(\theta)_n}$$

The propagation of stochastic monotonicity for sums $\beta_1 + \cdots + \beta_n$ is obvious, since $\beta_j$'s are two-valued. The regularity condition becomes $C(\pi_n) \sim \theta \log n$.

**Theorem** (AG–Pitman '05) Every Cayley-ISS virtual permutation is a mixture of Ewens' $P^\theta, \theta \in [0, \infty]$.