Space-time Brownian motion in an affine Weyl chamber and radial part of an Hermitian Brownian sheet.

M. Defosseux

IHP, feb. 23th 2017

1) Conditioned random walks and representation theory (Ph. Biane)

- Lie algebra $\mathfrak{sl}_2(\mathbb{C}) = \{M \in \mathcal{M}_2(\mathbb{C}) : \mathrm{tr}(M) = 0\}$
- For $x \in \mathbb{N}$, V_x : $x+1$ dimensional irreducible complex representation of $\mathfrak{sl}_2(\mathbb{C})$.
- Clebsch-Gordan rules :

$$V_x \otimes V_1 = V_{x+1} \oplus V_{x-1},$$

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  $x \in \mathbb{N}$ ($V_{-1} = \{0\}$).
- For $q \geq 1$, $x \in \mathbb{N}$, the character :

$$\mathrm{ch}_{V_x}(q) = q^x + q^{x-2} + \cdots + q^{-x} = \frac{q^{x+1} - q^{-(x+1)}}{q - q^{-1}} = s_x(q)$$

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$$s_x(q)s_1(q) = s_{x+1}(q) + s_{x-1}(q),$$

$$1 = \frac{s_{x+1}(q)}{s_x(q)s_1(q)} + \frac{s_{x-1}(q)}{s_x(q)s_1(q)}.$$

- A simple random walk (with drift) on $\mathbb{Z}$ conditioned to remain non negative, with a Markov kernel on $\mathbb{N}$

$$\widehat{K}(x,y) = \frac{s_y(q)}{s_x(q)s_1(q)} 1_{|y-x|=1}, \quad x, y \in \mathbb{N}.$$

- when $q = 1$

$$\widehat{K}(x,y) = \frac{y+1}{2(x+1)} 1_{|y-x|=1}, \quad x, y \in \mathbb{N}.$$

- For $x_0 \in \mathbb{N}$, $x \in \mathbb{N}$, Clebsch-Gordan rules :

$$s_x(q)s_{x_0}(q) = \sum_{y \in \mathbb{N}} m^y_{x,x_0} s_y(q).$$

- Markov kernel on $\mathbb{N}$

$$\widehat{K}(x,y) = \frac{s_y(q)}{s_x(q)s_{x_0}(q)} m^y_{x,x_0}, \quad x, y \in \mathbb{N}.$$

2) Conditioned Brownian motion

For $(\widehat{X}_n)$ the conditioned random walk, with $q = e^{\frac{\gamma}{\sqrt{n}}}$, $\gamma \geq 0$,

$$(\frac{\widehat{X}_{[nt]}}{\sqrt{n}}, t \geq 0) \underset{n\to\infty}{\longrightarrow} (\widehat{B}_t^{\gamma}, t \geq 0),$$

where $(\widehat{B}_t^{\gamma})$ is a Brownian motion with drift γ, conditioned to remain positive.

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where $(\widehat{B}_t^\gamma)$ is a Brownian motion with drift γ, conditioned to remain positive. Its transition densities $(\hat{p}_t)$ are

$$\hat{p}_t(x,y) = \frac{1 - e^{-2\gamma y}}{1 - e^{-2\gamma x}} p_t^0(x,y),$$
$$p_t^0(x,y) = p_t(x,y) - e^{-2\gamma x} p_t(-x,y), \quad x, y, t > 0.$$

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When $\gamma = 0$,

$$\hat{p}_t(x,y) = \frac{y}{x}(p_t(x,y) - p_t(x,-y)), \quad x, y, t > 0.$$

3) Affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$ (C. Lecouvey, E. Lesigne, M. Peigné)

- $\hat{\mathfrak{sl}}_2(\mathbb{C}) = \mathbb{C}[z, z^{-1}] \otimes \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $\mathbb{C}[z, z^{-1}]$ is the algebra of Laurent polynomials in z, + bracket.
- A Cartan subalgebra : $\mathfrak{h} = \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{C}c \oplus \mathbb{C}d$
- Weights :

$$P = \{x\Lambda_0 + y\frac{\alpha_1}{2} + z\delta : x, y \in \mathbb{N}\} \subset \mathfrak{h}^*$$

- Dominant weights :

$$P_+ = \{x\Lambda_0 + y\frac{\alpha_1}{2} + z\delta : 0 \leq y \leq x\} \cap P$$

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- Integrable highest-weight modules V_λ, $\lambda \in P_+$.
- Characters : for $\lambda \in P_+$, $\mathrm{ch}_\lambda = \sum_{\beta \in P} m_\lambda(\beta)e^\beta$.
- One has
$$\mathrm{ch}_\lambda(h) = \sum_{\beta \in P} m_\lambda(\beta)e^{\beta(h)} < \infty,$$
for $h = rd + \dots$, $r > 0$.

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$$\mathrm{ch}_\lambda(h)\mathrm{ch}_{\Lambda_0}(h) = \sum_{\beta \in P} m^{\beta}_{\lambda,\Lambda_0} \mathrm{ch}_\beta(h)$$

- Remark : $\lambda = x\Lambda_0 + \cdots \Rightarrow \beta = (x+1)\Lambda_0 + \dots$

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- Remark : $\lambda = x\Lambda_0 + \cdots \Rightarrow \beta = (x+1)\Lambda_0 + \ldots$
- Markov Kernel on P_+ :

$$\hat{K}(\lambda, \beta) = \frac{\mathrm{ch}_\beta(h)}{\mathrm{ch}_\lambda(h)\mathrm{ch}_{\Lambda_0}(h)} m^\beta_{\lambda,\Lambda_0}, \quad \lambda, \beta \in P_+$$

- Remark : $\widehat{X}_0 = 0$, $\widehat{X}_n = n\Lambda_0 + \ldots$.

1) A conditioned Space-Time Brownian motion
Convergence of the conditioned Markov chain $(\widehat{X}_{[nt]}, t \geq 0)$ when n goes to inifnity :

- For $h = \frac{1}{n^\beta} d$, $\beta + 1 - 2\alpha = 0$, $\alpha \in [1/2, 1[$,

$$\frac{1}{n^\alpha}\widehat{X}_{[nt]} \xrightarrow[n\to\infty]{d} +\infty\Lambda_0 + \hat{x}_t^+ \frac{\alpha_1}{2} \mod \delta$$

- For $h = \frac{1}{n} d$,

$$\frac{1}{n}\widehat{X}_{[nt]} \xrightarrow[n\to\infty]{d} t\Lambda_0 + \hat{x}_t^a \frac{\alpha_1}{2} \mod \delta$$

- For $h = \frac{1}{n^\beta} d$, $\beta + 1 - 2\alpha = 0$, $\alpha > 1$, $(\widehat{X}_0 = n^\alpha \Lambda_0)$,

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1) Radial part

- $SU(2) = \{M \in \mathcal{M}_2(\mathbb{C}) \, : \, MM^* = I, \, \det(M) = 1\}$
- $\mathfrak{su}(2) = \{M \in \mathcal{M}_2(\mathbb{C}) \, : \, M + M^* = 0, \, \mathrm{tr}(M) = 0\}$

$$= \{M = \begin{pmatrix} ix & iy - z \\ iy + z & -ix \end{pmatrix}, \, x, y, z \in \mathbb{R}\}$$

- Adjoint action : $Ad(k)M = kMk^*, k \in SU(2), \; M \in \mathfrak{su}(2)$
- $\forall M \in \mathfrak{su}(2) \, \exists! r \in \mathbb{R}_+ \, : M = k \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix} k^*$, for some $k \in SU(2)$.
- Radial part : $\mathrm{rad}(M) = \begin{pmatrix} ir & 0 \\ 0 & -ir \end{pmatrix}, \, r = \sqrt{x^2 + y^2 + z^2}$.

2) Radial part of a Brownian motion on $\mathfrak{su}(2)$

- A Brownian motion on $\mathfrak{su}(2)$: $b_t = \begin{pmatrix} ix_t & iy_t - z_t \\ iy_t + z_t & -ix_t \end{pmatrix}$, $t \geq 0$
- The radial part process :
$$r_t = \begin{pmatrix} i\sqrt{x_t^2 + y_t^2 + z_t^2} & 0 \\ 0 & -i\sqrt{x_t^2 + y_t^2 + z_t^2} \end{pmatrix}, t \geq 0.$$
- $(\sqrt{x_t^2 + y_t^2 + z_t^2}, t \geq 0)$: Brownian motion conditioned to remain positive

3) What did we do ?

- $K = SU(2)$ a compact Lie group, $\mathfrak{k} = \mathfrak{su}(2)$ its Lie algebra.
- $\mathfrak{h} = \{\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} : x \in \mathbb{R}\}$, a Cartan subalgebra.
- An adjoint action $Ad : K \to GL(\mathfrak{k})$
- $W = \mathfrak{k}/Ad(K) = \{\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} : x \geq 0\}$, a Weyl chamber
- Equipp $\mathfrak{k}$ with a scalar product $(M, N) = \frac{1}{2}\mathrm{tr}(MN)$.

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$(b_t)_{t\geq 0}$ Brownian motion on $\mathfrak{su}(2)$ $\longrightarrow$ $(r_t)_{t\geq 0}$ radial part process on W.

$\downarrow$ projection of $(b_t)_{t\geq 0}$ on $\mathfrak{h}$ $\qquad\qquad$ $\overset{d}{=}$

$(x_t)_{t\geq 0}$ Brownian motion on $\mathfrak{h}$ $\longrightarrow$ $(x_t)_{t\geq 0}$ conditioned to remain in W

III-Loop groups and Loop Algebras : Coadjoint orbit and Radial part

(Pressley, Segal)

1) Affine Lie algebra and Coadjoint action of the loop group.

- $\mathcal{L}SU(2) = \{f : [0,1] \to SU(2) : f(0) = f(1)\}$ + regularities
- $\mathcal{L}\mathfrak{su}(2) = \{f : [0,1] \to \mathfrak{su}(2) : f(0) = f(1)\}$ + regularities.
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- scalar product $(.,.)$ on $\mathfrak{su}(2)$.
- (Real) Affine Lie algebra $\mathcal{L}\mathfrak{su}(2) \oplus \mathbb{R}c$, with Lie bracket

$$[\xi + \lambda c, \eta + \mu c] = [\xi, \eta] + w(\xi, \eta)c,$$

$w(\xi, \eta) = \int_0^1 (\xi'(t), \eta(t))\, dt.$

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$$[\xi + \lambda c, \eta + \mu c] = [\xi, \eta] + w(\xi, \eta)c,$$

  $w(\xi, \eta) = \int_0^1 (\xi'(t), \eta(t))\, dt.$
- Coadjoint action of $\mathcal{L}SU(2)$ on $\mathcal{L}\mathfrak{su}(2)' \oplus \mathbb{R}\Lambda_0$ ($\Lambda_0(c) = 1$, $\Lambda_0(\mathcal{L}\mathfrak{su}(2)) = 0$) :

$$\gamma.(\phi + \lambda\Lambda_0) = [Ad^*(\gamma)\phi - \lambda \int_0^1 (\gamma_s'\gamma_s^{-1}, .)\, ds] + \lambda\Lambda_0,$$

  where $\gamma \in \mathcal{L}SU(2)$, $\phi \in \mathcal{L}\mathfrak{su}(2)'$, $x \in \mathcal{L}\mathfrak{su}(2)$, $\lambda \in \mathbb{R}$, and $Ad^*(\gamma)\phi(x) = \phi(\gamma^{-1}x\gamma)$.

2) Coadjoint orbit and Radial part
Now, $\mathcal{L}\mathfrak{su}(2)$ is the completion of the previous one equipped with the L_2 norm.
Consider the Cameron-Martin space

$$H^1 = \{x : [0,1] \to \mathfrak{su}(2) : x(0) = 0, \dot{x} \in L_2\}$$

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For $x \in H^1$, given $\lambda > 0$, the action of $\mathcal{L}SU(2)$ on $\phi_x \in \mathcal{L}\mathfrak{su}(2)'$, defined by

$$\phi_x(y) = \int_0^1 (y_s, \dot{x}_s)\, ds, \quad y \in L_2$$

is given by

$$\gamma.(\phi_x + \lambda\Lambda_0) = \int_0^1 (., \gamma_s \dot{x}_s \gamma_s^{-1} - \lambda \gamma_s' \gamma_s^{-1})\, ds + \lambda\Lambda_0,$$

for $\gamma \in \mathcal{L}SU(2)$, $\gamma'\gamma^{-1} \in L_2$.

$\mathcal{L}\mathfrak{su}(2)' \oplus \mathbb{R}\Lambda_0 \sim H^1 \oplus \mathbb{R}\Lambda_0$

For $\lambda > 0$, $x \in H^1$, denotes by $\epsilon(x + \lambda\Lambda_0)$ the solution of

$$\lambda\, d\epsilon(x + \lambda\Lambda_0) = \epsilon(x + \lambda\Lambda_0)\, dx,$$

with initial condition $\epsilon(x + \lambda\Lambda_0)_0 = e$.

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Proposition

The linear form $\phi_x + \lambda\Lambda_0$ and $\phi_y + \lambda\Lambda_0$ are in the same orbit for the action of $\mathcal{L}SU(2)$ if and only if the endpoint $\epsilon(x + \lambda\Lambda_0)_1$ and $\epsilon(y + \lambda\Lambda_0)_1$ are in the same orbit for the adjoint action of $SU(2)$ on itself.

Definition

For $\lambda > 0$, $x \in H^1$, *one defines the radial part of* $\phi_x + \lambda\Lambda_0$ *as the linear form* $\phi_{\pi_r} + \lambda\Lambda_0$, *where where* $\pi_r(t) = t\begin{pmatrix} i\pi r & 0 \\ 0 & -i\pi r \end{pmatrix}$, $t \in [0,1]$, *and* r *is the unique real number in* $[0,\lambda]$ *such that*

$$\epsilon(x + \lambda\Lambda_0)_1 = k\begin{pmatrix} e^{i\pi\frac{r}{\lambda}} & 0 \\ 0 & e^{-i\pi\frac{r}{\lambda}} \end{pmatrix} k^*,$$

for some $k \in SU(2)$. *It is denoted by* $\mathrm{rad}(\phi_x + \lambda\Lambda_0)$ *or* $\mathrm{rad}(x + \lambda\Lambda_0)$.

3) Restriction to a Cartan sub-algebra of $\mathcal{L}\mathfrak{su}(2) \oplus \mathbb{R}c$.

- A Cartan sub-algebra $\sim \mathfrak{h} \oplus \mathbb{R}c$.
- $(A, B) = \frac{1}{(2\pi)^2}\mathrm{tr}(AB^*)$, $A, B \in \mathfrak{su}(2)$.
- $\alpha_1 \in \mathfrak{h}^*$ defined by $\alpha_1(H_u) = \frac{2u}{2\pi}$, for $H_u = \begin{pmatrix} iu & 0 \\ 0 & -iu \end{pmatrix}$.

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For $\lambda > 0$, $r \in [0, \lambda]$, one has

$$(\phi_{\pi_r} + \lambda\Lambda_0)_{|\mathfrak{h}+\mathbb{R}c} = r\frac{\alpha_1}{2} + \lambda\Lambda_0 \in W^{aff},$$

where $W^{aff} = \{\lambda\Lambda_0 + r\frac{\alpha_1}{2} : 0 \leq r \leq \lambda\} \subset \mathfrak{h}^* \oplus \mathbb{R}\Lambda_0$.

(I. Frenkel)

Definition

For $\lambda > 0$, and $x = (x_s)_{s\in[0,1]}$ a $\mathfrak{su}(2)$-valued Brownian motion, one defines the radial part of $(x + \lambda\Lambda_0)$ as the unique real number r in $[0, \lambda]$ such that

$$\epsilon(x + \lambda\Lambda_0)_1 = k \begin{pmatrix} e^{i\pi\frac{r}{\lambda}} & 0 \\ 0 & e^{-i\pi\frac{r}{\lambda}} \end{pmatrix} k^*,$$

for some $k \in SU(2)$, where $\epsilon(x + \lambda\Lambda_0)_1$ is the solution of the SDE

$$\lambda\, d\epsilon(x + \lambda\Lambda_0) = \epsilon(x + \lambda\Lambda_0) \circ dx,$$

with initial condition $\epsilon(x + \lambda\Lambda_0)_0 = e$. It is denoted by $\mathrm{rad}(x + \lambda\Lambda_0)$.

Proposition

Let $(x_s^t)_{s\in[0,1],t\in\mathbb{R}_+}$ *be a standard Brownian sheet on* $\mathfrak{su}(2)$*. Then*

$$t\Lambda_0 + rad(x^t + t\Lambda_0)\frac{\alpha_1}{2}, t > 0,$$

is a space-time Brownian motion

$$t\Lambda_0 + b_t\frac{\alpha_1}{2} = t\Lambda_0 + (x_1^t, .)_{|\mathfrak{h}}, t > 0,$$

conditioned to remain in the affine Weyl chamber W^{aff}*.*

What did we do ?

" Brownian motion on $\mathbb{R}\Lambda_0 \oplus \mathcal{L}\mathfrak{su}(2)'$ " $t\Lambda_0 + \int_0^1 (., dx_s^t), t \geq 0$	$\longrightarrow$	Radial part process on W^{aff}. $(t\Lambda_0 + r_t \frac{\alpha_1}{2}), t \geq 0.$
$\downarrow$ restriction to $\mathfrak{h} \oplus \mathbb{R}c$		$\stackrel{d}{=}$
Brownian motion on $\mathbb{R}\Lambda_0 \oplus \mathfrak{h}^*$ $t\Lambda_0 + (x_1^t, .)_{\|\mathfrak{h}}, \ t \geq 0.$	$\longrightarrow$	Brownian motion on $\mathbb{R}\Lambda_0 \oplus \mathfrak{h}^*$ conditioned to remain in W^{aff}