Lecture 1: Integrable probabilities

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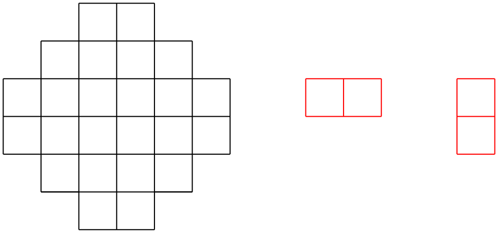
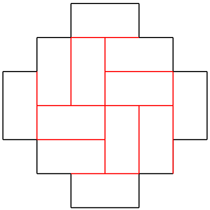
- Lecture 1: Uniformly random domino tilings of the Aztec diamond
- Lectures 1-2: Algebra of symmetric functions; Schur processes.
- Lectures 2-3: RSK-algorithm; Last Passage Percolation.
- Lecture 3: Stochastic six vertex model; Hall-Littlewood processes.
- Lectures 4-5: Global asymptotics of stochastic particle systems: robust results.
- Lectures 6-7: Determinantal processes; correlation functions for Schur processes; local asymptotic limit behavior.
- Lecture 8: Asymptotics of the stochastic six vertex model.

Domino tilings of Aztec diamond

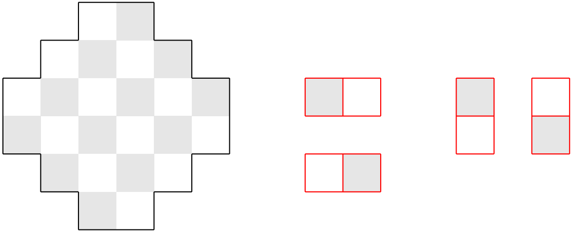
Aztec diamond of size 3.



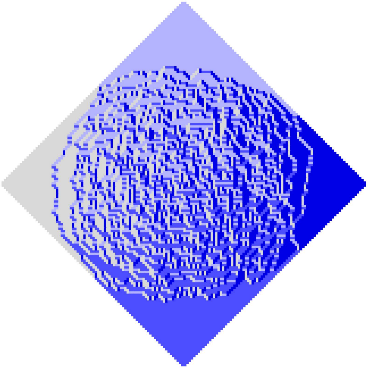


- Aztec diamond of size n is all lattice squares which are (fully) contained in $\{(x, y) : |x| + |y| \leq n + 1\}$.
- Domino tilings of Aztec diamond were introduced by Elkies-Kuperberg-Larsen-Propp'92. They proved that the number of tilings is equal to $2^{n(n+1)/2}$.
- Exercise: Check that Aztec diamond of size 2 has 8 domino tilings.
- Question: What happens when we consider a uniformly random domino tiling of a large Aztec diamond ?

Let us consider a chessboard coloring of Aztec diamond. It is useful to distinguish not two, but four different types of dominoes.



We will color these four types by different colors in the next picture.

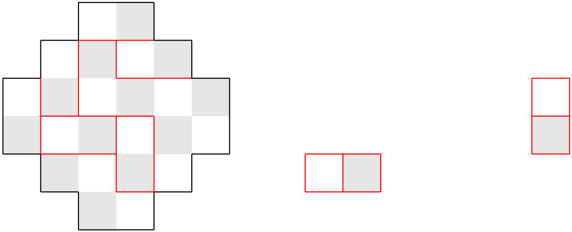


We see that a uniformly random domino tiling has some structure !

Theorem (Jockush-Propp-Schor'98): Asymptotically a uniformly random tiling becomes frozen outside of a certain circle.

There are many more interesting properties of these tilings.

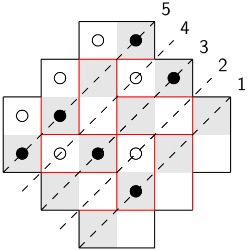
Let us consider dominoes of two types only:

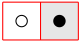


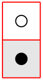
Lemma: Each domino tiling is uniquely determined by dominoes of two types.

Idea of proof: Moving from north-west to south-east, we see that all missing dominoes can be reconstructed in a unique way.

(Johansson'02, '05) Let us put particles of two different types:







On level 0 we have 0 particles, then on levels 1 and 2 we have 1 particle, on levels 3 and 4 we have 2 particles, on levels 5 and 6 we have three particles.

Now we want to understand when a particle system corresponds to a domino tiling of the Aztec diamond.

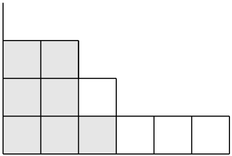
A non-increasing sequence of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ is called a *signature* of length N.

We say that a signature λ of length N and signature μ of length $N-1$ *interlace* if

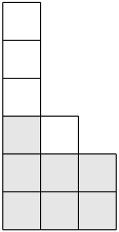
$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{N-1} \geq \lambda_N.$$

Notation $\mu \prec \lambda$. Example: $(3,3,2) \prec (6,3,2,0)$.

Geometrically, they differ by a *horisontal strip* (no two boxes in the same column).



We say that signatures λ and μ of length N differ by a *vertical strip* if $\lambda_i - \mu_i = \{0, 1\}$, for any $1 \leq i \leq N$.
Notation $\mu \prec_v \lambda$.
For example, $(3, 3, 1, 0, 0, 0) \prec_v (3, 3, 2, 1, 1, 1)$



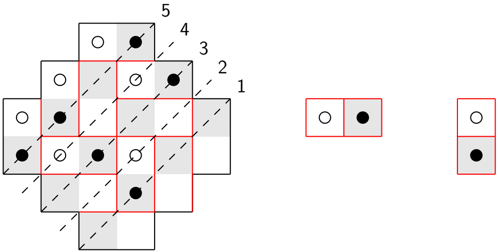
Note that this example is obtained by symmetry with respect to $x = y$ from the previous one. This is a *transposition* of signatures / Young diagrams.

If λ is a signature of length N, then $\{\lambda_i + N - i\}_{i=1}^{N}$ is a collection of distinct integers. In the opposite direction, given a collection of distinct integers, one can construct a signature.

Consider all collections of signatures with nonnegative coordinates $\lambda^1, \ldots, \lambda^N, \mu^1, \ldots, \mu^{N-1}$, such that λ^i, μ^i have length i and

$$\emptyset \prec \lambda^1 \succ_v \mu^1 \prec \lambda^2 \succ_v \mu^2 \prec \cdots \prec \lambda^N \succ_v (0, 0, \ldots, 0)$$

Proposition. The set of collection of signatures is in bijection with domino tilings of Aztec diamond.



On each level boxes are numbered starting from 0. Distinct particles are in positions:

$$1; 1; (1,3); (0,2); (0,1,3); (0,1,2);$$

Making shifts, we obtain signatures:

$$\emptyset \prec 1 \succ_v 1 \prec (1,2) \succ_v (0,1) \prec (0,0,1) \succ_v (0,0,0)$$

Exercise: do this construction for a couple of domino tilings of Aztec diamond of size 3.
Condition:

$$\emptyset \prec \lambda^1 \succ_v \mu^1 \prec \lambda^2 \succ_v \mu^2 \prec \cdots \prec \lambda^N \succ_v (0,0,\ldots,0)$$

Proposition. The set of collections of signatures $(\lambda^1, \mu^1, \ldots, \lambda^N)$ is in bijection with domino tilings of Aztec diamond.

Idea of proof: They correspond to black and white particles. Moving from south-east to north-west, one can check that the conditions on dominoes and on signatures coincide.

— How to compute the number of signatures with such conditions ?

— How to analyze the properties of uniformly random collection of signatures ?

Tools: Symmetric functions. Schur processes.